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PENNY-SHAPED CRACK
IN AN ELASTIC-PLASTIC SOLID

BY

MOHAMMAD HOSSEIN SADEGHI


A thesis submitted
in partial fulfillment of the requirements for the
degree Master of Science, Major in
Mechanical Engineering, South
Dakota State University

1969

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PENNY-SHAPED CRACK
IN AN ELASTIC-PLASTIC SOLID

This thesis is approved as a creditable and independent investigation by a candidate for the degree, Master of Science, and is acceptable as meeting the thesis requirements for this degree, but without implying that the conclusions reached by the candidate are necessarily the conclusions of the major department.



Thesis Adviser

Date



Head, Mechanical Engineering
Department

Date

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MHS

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LIST OF SYMBOLS

Symbol	Meaning
r, θ, z	Cylindrical coordinate, (r, θ) two dimensional problem
${}^0 \varepsilon_r, {}^0 \varepsilon_\theta, {}^0 \varepsilon_z$	Purely elastic normal strains
${}^0 \gamma_{r\theta}, {}^0 \gamma_{\theta z}, {}^0 \gamma_{zr}$	Purely elastic shear strains
${}^0 \sigma_r, {}^0 \sigma_\theta, {}^0 \sigma_z$	Purely elastic normal stresses
${}^0 \tau_{r\theta}, {}^0 \tau_{\theta z}, {}^0 \tau_{zr}$	Purely elastic shear stresses
E, G	Young's modulus and shear modulus
ν	Poisson's ratio
r	Radius
p_0	Pressure
u_r, u_θ, u_z	Displacement in r, θ , and z directions respectively
${}^e \varepsilon_r, {}^e \varepsilon_\theta, {}^e \varepsilon_z$	Elastic normal strains
${}^e \gamma_{r\theta}, {}^e \gamma_{\theta z}, {}^e \gamma_{zr}$	Elastic shear strains
${}^p \varepsilon_r, {}^p \varepsilon_\theta, {}^p \varepsilon_z$	Plastic normal strains
${}^p \gamma_{r\theta}, {}^p \gamma_{\theta z}, {}^p \gamma_{zr}$	Plastic shear strains
α	Unknown parameter
${}^e \sigma_r, {}^e \sigma_\theta, {}^e \sigma_z$	Elastic normal stresses
${}^e \tau_{r\theta}, {}^e \tau_{\theta z}, {}^e \tau_{zr}$	Elastic shear stresses
${}^p \sigma_r, {}^p \sigma_\theta, {}^p \sigma_z$	Plastic normal stresses
${}^p \tau_{r\theta}, {}^p \tau_{\theta z}, {}^p \tau_{zr}$	Plastic shear stresses
σ_1	Stress intensity

ε_1	Strain intensity
Y	Yield stress at uniaxial tension
e_u, p_u	Density of strain energy in elastic and plastic regions respectively
U	Total strain energy
W	Work of external forces on the system
I^*	Potential energy of the system
l	Half of major axis of the crack
δ	Distance measured from the tip of the crack
ψ	Angle measured at the tip of the crack
λ	Dimensionless load ($\lambda = p_0/Y$)
φ	Generalized plastic modulus
σ_1	Maximum principal stress
σ_m	Medium stress
D	Energy dissipation
w	Dimensionless displacement at the center of crack
$P_{crit.}$	Critical stress
SE	Surface energy
γ	Specific surface energy
ξ	Dimensionless crack length
$\bar{\sigma}_z, \bar{\sigma}_r, \bar{\sigma}_\theta, \bar{\tau}_{rz}$	Normal stresses and shear stress of superposed state

CHAPTER I

INTRODUCTION

The field of fracture mechanics has been of great interest to scientists for many years. In 1776, Coloumb [9], a French scientist, proposed a certain criterion for yielding. This idea was later developed by an Italian researcher Tresca. More recently, Weibull [9], a German scientist, studied the statistical aspect of fracture.

In 1913, Inglis [8] derived the mathematical formulae for the stress distribution around an elliptical void, and has demonstrated that for a very narrow ellipse (say crack) the stress concentration factor becomes very large. In 1921, A. A. Griffith [4] and [5], a British scientist, was led by this idea to propose his theory of rupture and flow in solids. It was Griffith who opened the field of fracture mechanics for the contemporary scientists.

Griffith has used the equations describing the stress distribution around a crack, as derived earlier by Inglis, to derive a theoretical formula for the critical stress. Although this equation was not scientifically recognized at that time, it is of a great practical importance today. The first American scientist who contributed to fracture mechanics is Westergaard [18]. He derived the equations for stress

distribution around a two dimensional crack by means of a stress function of complex variable.

Two of the more recent researchers who contributed substantially to the field of fracture mechanics are I. N. Sneddon [16] and R. Sack [15] who are both British. These scientists have both derived expressions for the critical stress around a three dimensional penny-shaped crack. In addition, Sneddon found the equations for stress distribution around a penny-shaped crack [16], and succeeded in expressing them in an asymptotic form, i.e. when the distance from the crack tip is small. Like all the scientists mentioned here, Sneddon and Sack also treat the crack problems assuming that the material surrounding the crack is an ideally elastic solid.

In the recent years a considerable attention has been given to certain non-elastic problems of fracture mechanics involving plastic and visco-elastic solids. Let us mention here the monographs by A. S. Tetelman and A. J. McEvily [17]. These authors have discussed the macroscopic and microscopic aspects of fracture, and they have also discussed a formidable amount of experimental data obtained for non-elastic solids, but they have never attempted to analyze the problem mathematically.

A. Mendelson [11] has discussed the various plasticity theorems as applied to fracture problems in his book published recently; but he has not introduced new ideas except

for a careful report on the works of the other scientists.

G. R. Irwin [10] and E. Orowan [13] have modified the Griffith's theory basing it on the energy considerations; they have independently showed that the critical stress is not entirely dependent on the specific surface energy as it was assumed by Griffith, but rather it is controlled by the plastic dissipation energy of the crack tip. For example, the critical stress for ductile steel differs from what could be obtained from Griffith's equation because, the plastic dissipation energy for ductile steel is about $2 \times 10^6 \text{ erg/cm}^2$ while the specific surface energy is only about 10^3 erg/cm^2 .

Up to now no one has obtained the exact solution for a crack in an elastic-plastic solid except for J. Hult and F. C. McClintock [6] who have solved exactly the problem of propagating crack in an ideally elastic-plastic solid for anti-plane shear. Two of the recent works have been done by J. R. Rice and G. F. Rosengren [14], and J. W. Hutchinson [7] who have obtained a "near field" solution (i.e. true in an asymptotic case) to the stress and strain distribution around the crack tip.

Growth of plastic zone from the tips of cracks or notches has been a subject of investigation for years. Some tried to analyze this problem by introducing certain models such as, those of Dugdale [2], Barenblatt [1], and Wnuk [21]. Dugdale led by an ingenious intuition, proposed that (at least for plane-stress condition) the plastic deformation at the crack

tip is entirely confined to a narrow tapered zone. G. I. Barenblatt [1] has introduced a modulus called "cohesive modulus" in order to explain the quasi-ductile fracture (in fact he meant brittle solids, but it has been established that his model is also very useful for ductile fracture if only the plastic zone is localized).

Dugdale's idea has been used by J. N. Goodier and F. A. Field [3] to explain the shape of the plastic zone in two dimensional case (plane stress and plane strain). Olesiak and Wnuk [12] have obtained a three dimensional solution for the shape of the plastic zone by the use of Dugdale's concept. Also Wnuk [19] and [20] has worked on a problem which concerns the nature of the fracture in plastic and visco-elastic solids extending the energy criterion to non-elastic cases and employing the Dugdale model.

In chapter two of this thesis we determine the approximate stress and strain distribution in an ideally elastic-plastic solid containing a pressurized circular hole by the use of variational principle. We also calculate the critical stress at which the plastic zone will start to appear.

Chapter three of this thesis deals with the distribution of stress and strain around a pressurized penny-shaped crack by the use of variational principle and Sneddon's [16] asymptotic equations. Elastic strains are evaluated from elastic stresses by means of the Hooke's law and certain basic assumptions. The plastic strains are assumed to be equal to the

elastic strains which are scaled by a certain priori unknown parameter. Employing the Hencky-Ilyushin constitutive relations for the plastic zone, we obtain the stresses within the plastic zone surrounding the penny-shaped crack. Next applying the Huber-Mises-Hencky plasticity condition [11], we predict the elastic-plastic boundary.

The distribution of normal stresses in elastic and plastic regions are compared with Sneddon's solution which has been obtained for a purely elastic solid. Stress intensity $\sigma_1 = \sqrt{3/2} \sqrt{S_{1j} S_{1j}}$ in elastic region is larger than purely elastic stress intensity, but within the plastic zone the plastic stress intensity is finite while the elastic counterpart tends toward infinity as $\delta \rightarrow 0$, where δ denotes the distance measured from the crack tip.

The remainder of chapter three is devoted to investigation of different forms of elastic-plastic boundary by postulating a more general form of the plasticity condition. The effect of distortion and volume energies on the shape of elastic-plastic boundary has been studied, and the conclusion shows that the distortion energy has more influence on the shape of the elastic-plastic boundary than the volume energy. Also, we have established that the volume and distortional energies entering a generalized plasticity condition cannot possess any arbitrary cofactors, but the cofactors must satisfy a certain inequality which is dependent on the Poisson's ratio of the solid.

The contours of equal reduced strain are graphed, and it is found that the shape of the elastic-plastic boundary changes considerably upon assuming different shares of shear intensity and dilatational strain. At the end of chapter three, we investigate the shape of the plastic zone based on the maximum principal stress criterion for yielding which may be justified better than the Huber-Mises-Hencky criterion for solids exhibiting a fiber-like structure (e.g. polymers). The pattern of plastic zone obtained seems to approach the shape of the plastic zone assumed in the Barenblatt-Dugdale model of the crack.

In chapter four part A, we have evaluated the strain energies in the elastic and plastic regions and the total strain energy of the solid. The ratio of the plastic strain energy to the total strain energy is found to be small as long as the ratio of the applied load to the yield stress is small (our theory does not permit to evaluate these quantities for larger values of the applied load). Finally, since only distortional energy can be dissipated, it is shown that dissipation energy released during the yielding process is slightly less than the strain energy contained within the plastic zone.

In part B of chapter four, the principle of minimum potential energy is applied to evaluate the unknown parameter α . The cubic equation containing the parameter α is solved by approximation method. It is found that α is slightly

larger than unity and its value depends on the Poisson's ratio as well as the dimensionless load λ .

The part C of chapter four contains the relation between the applied stress and the displacement at the center of the crack. The displacements on the surface of the crack and at the center of the crack for purely elastic solid and for an elastic-plastic solid are discussed. Equations and graphs show that the purely elastic displacement is a linear function of the applied stress, but the elastic-plastic displacement is a polynomial and thus a non-linear function of the applied stress and it is slightly larger than the purely elastic displacement.

The critical stress is evaluated in chapter five by means of the minimum energy theorem. The critical stress is found to be smaller than the one obtained by Sack and Sneddon [16] for a purely elastic solid and modified according to Irwin-Orowan modification. Our solution for the critical stress is only approximate and when the characteristic length of crack reaches a certain value $\xi=0.039$, the critical stress for the elastic-plastic solid becomes a constant; while for a purely elastic solid the critical stress approaches infinity.

In the final chapter, we have attempted to solve the case where load is applied remotely from the crack. Figure 13 shows the difference between the size of the plastic zone for this case and the one obtained in chapter three.

CHAPTER II

EXAMPLE OF APPLICATION OF VARIATIONAL PRINCIPLE TO AN ELASTIC-PLASTIC PROBLEM

To illustrate the application of the variational principle for solving an elastic-plastic problem, we will consider the stress and strain distributions around a circular hole which is subjected to constant internal pressure p_0 . We assume an ideally homogeneous, incompressible, isotropic, elastic-plastic solid and the state of plane strain ($\epsilon_z = \gamma_{z\theta} = \gamma_{rz} = 0$). The plane (r, θ) is divided into two sub-regions

- | | | |
|-----|-----------------|---------------------|
| (1) | Plastic region. | $a \leq r \leq r^*$ |
| (2) | Elastic region. | $r > r^*$ |

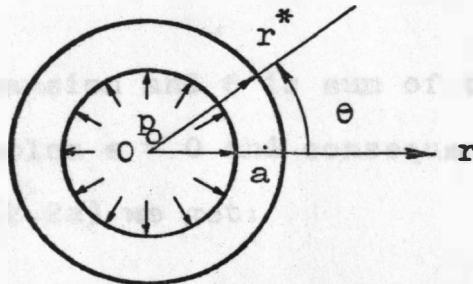


Figure 2.1

where a is the radius of the hole, r^* denotes the radius of the primarily unknown plastic region.

The problem is first solved for the purely elastic state by the use of the polar coordinate system. The elastic-plastic case is then solved by the use of certain assumptions and by applying the variational approach.

Due to axial symmetry and absence of body force, the general equations of equilibrium reduce to

$$(2.1) \quad \frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0.$$

From Hooke's law it can be shown that

$$(2.2a) \quad \epsilon_r + \epsilon_\theta + \epsilon_z = \frac{1-2\nu}{E} (\sigma_r + \sigma_\theta + \sigma_z)$$

or that

$$(2.2b) \quad e = \frac{1-2\nu}{E} \theta$$

where e is volume expansion and θ is sum of the normal stresses. In this problem $e = 0$ and consequently $\theta = 0$; hence from equation (2.2a) we get:

$$(2.3) \quad \epsilon_r + \epsilon_\theta = 0$$

and therefore

$$^{\circ}\varepsilon_r = -^{\circ}\varepsilon_{\theta}.$$

Also

$$^{\circ}\varepsilon_z = \frac{1}{E} \left[^{\circ}\sigma_z - \nu (^{\circ}\sigma_r + ^{\circ}\sigma_{\theta}) \right] = 0$$

which reduces to

$$^{\circ}\sigma_z = \nu (^{\circ}\sigma_r + ^{\circ}\sigma_{\theta}).$$

Now from equation (2.2b) we have

$$0 = \theta = ^{\circ}\sigma_r + ^{\circ}\sigma_{\theta} + \nu (^{\circ}\sigma_r + ^{\circ}\sigma_{\theta}) = (1 + \nu) (^{\circ}\sigma_r + ^{\circ}\sigma_{\theta}).$$

Since in our case $\nu = 1/2$, we must have

$$^{\circ}\sigma_r + ^{\circ}\sigma_{\theta} = 0$$

or

$$(2.3) \quad ^{\circ}\sigma_{\theta} = -^{\circ}\sigma_r.$$

Substituting $-^{\circ}\sigma_r$ for $^{\circ}\sigma_{\theta}$ from (2.3) into equation (2.1) we get:

$$\frac{d^{\circ}\sigma_r}{dr} + \frac{2^{\circ}\sigma_r}{r} = 0$$

Solving this ordinary differential equation for ${}^o\sigma_r$ we obtain

$${}^o\sigma_r = \frac{1}{r^2 C}$$

where C is an arbitrary constant. At the boundary where $r = a$ and ${}^o\sigma_r = -p_o$; C is determined by a boundary condition for ${}^o\sigma_r$. Hence

$$\frac{1}{a^2 C} = -p_o$$

and therefore

$$C = -1/a^2 p_o ;$$

the stresses then become

$$(2.4) \quad {}^o\sigma_r = -p_o \left[\frac{a}{r} \right]^2$$

$${}^o\sigma_\theta = p_o \left[\frac{a}{r} \right]^2$$

The elastic stress distribution is illustrated in figure 2.2.

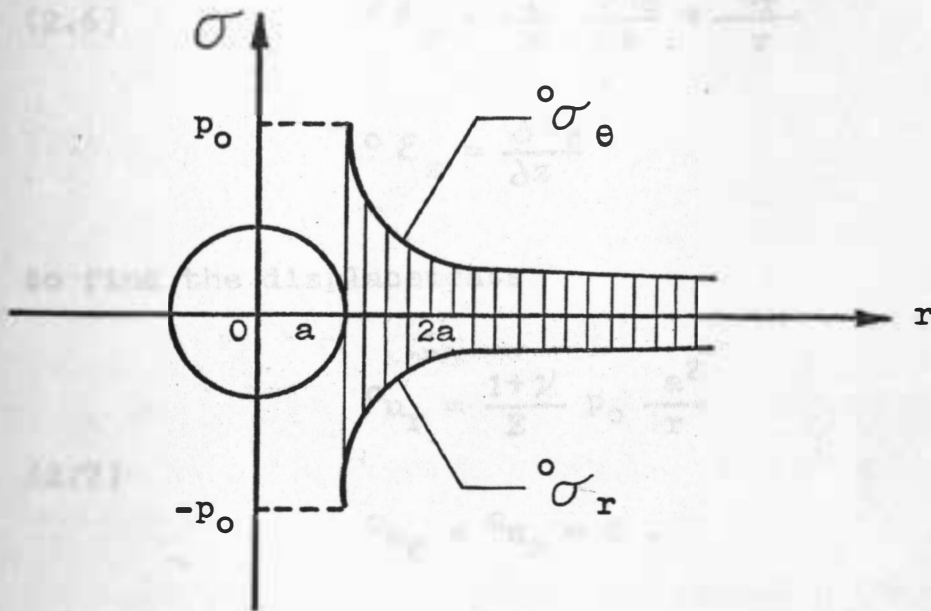


Figure 2.2

We use Hooke's law to find the strains

$$\begin{aligned}
 \epsilon_r &= -\frac{1+\nu}{E} p_0 \left[\frac{a}{r} \right]^2 \\
 \epsilon_\theta &= \frac{1+\nu}{E} p_0 \left[\frac{a}{r} \right]^2 \\
 \epsilon_z &= 0
 \end{aligned}
 \tag{2.5}$$

and we use the strain-displacement relations in the cylindrical coordinate system

$$\epsilon_r = \frac{\partial u_r}{\partial r}$$

$$(2.6) \quad {}^0\varepsilon_{\theta} = \frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta} + \frac{u_r}{r}$$

$${}^0\varepsilon_z = \frac{\partial u_z}{\partial z}$$

to find the displacements

$$(2.7) \quad {}^0u_r = \frac{1+\nu}{E} p_0 \frac{a^2}{r}$$

$${}^0u_{\theta} = {}^0u_z = 0.$$

In order to solve for elastic-plastic stresses, we assume that the elastic-plastic strains are obtained from the purely elastic strains by multiplying the latter by a certain unknown coefficient α . The unknown parameter α will be determined by the variational principle.

We assume

$${}^e\varepsilon_r = {}^p\varepsilon_r = \alpha {}^0\varepsilon_r$$

$$(2.8) \quad {}^e\varepsilon_{\theta} = {}^p\varepsilon_{\theta} = \alpha {}^0\varepsilon_{\theta}$$

$${}^e\varepsilon_z = \alpha {}^0\varepsilon_z = 0$$

and

$${}^e\sigma_r = \alpha^0 \sigma_r = -\alpha p_0 \left(\frac{a}{r}\right)^2$$

$$(2.9) \quad {}^e\sigma_\theta = \alpha^0 \sigma_\theta = \alpha p_0 \left(\frac{a}{r}\right)^2$$

$${}^e\sigma_z = 0.$$

Due to axial symmetry about an axis passing through the point 0 and perpendicular to the plane (r, θ) , we have ${}^0\tau_{r\theta} = 0$.

Now we apply the Huber-Mises-Hencky plasticity condition to the boundary between the elastic and plastic regions. We first evaluate the stress intensity ${}^e\sigma_1$ which is given by the following relationship

$$(2.10) \quad {}^e\sigma_1 = \sqrt{\frac{3}{2}} \sqrt{{}^e s_{ij} {}^e s_{ij}} =$$

$$\frac{1}{\sqrt{2}} \left[({}^e\sigma_r - {}^e\sigma_z)^2 + ({}^e\sigma_z - {}^e\sigma_\theta)^2 + ({}^e\sigma_\theta - {}^e\sigma_r)^2 + \right.$$

$$\left. 6({}^e\tau_{rz}^2 + {}^e\tau_{r\theta}^2 + {}^e\tau_{\theta z}^2) \right]^{\frac{1}{2}}.$$

Substituting the proper values from equations (2.9) into equation (2.10) and equating ${}^e\sigma_1$ to the yield stress Y as required by the Huber-Mises-Hencky theory gives

$$(2.11a) \quad {}^e\sigma_1 = Y = \sqrt{3} p_0 \alpha \left(\frac{a}{r^*}\right)^2$$

which results in

$$(2.11b) \quad r^{*2} = \frac{\sqrt{3} p_0 \alpha}{Y} a^2 .$$

This indicates that the elastic-plastic boundary is a circle as indicated in figure 2.1 . For simplicity we denote the term $\lambda = \frac{\sqrt{3} p_0}{Y}$; thus the equation (2.11b) becomes:

$$(2.12) \quad r^* = \sqrt{\alpha \lambda} a .$$

In solving for α the principle of minimum potential energy is used. This principle states that the system is in equilibrium when the potential energy of the system is minimum. The potential energy of the system is given by the difference of the elastic strain energy and the change in the potential energy of the external forces. For linear elastic solids the latter equals twice the work done by the external forces acting on the system. Let us denote the potential energy by I^* , then:

$$(2.13) \quad I^* = U - W$$

where

$$U = \int_V e u dv + \int_V p u dv ; W = \int_S T_i u_i ds .$$

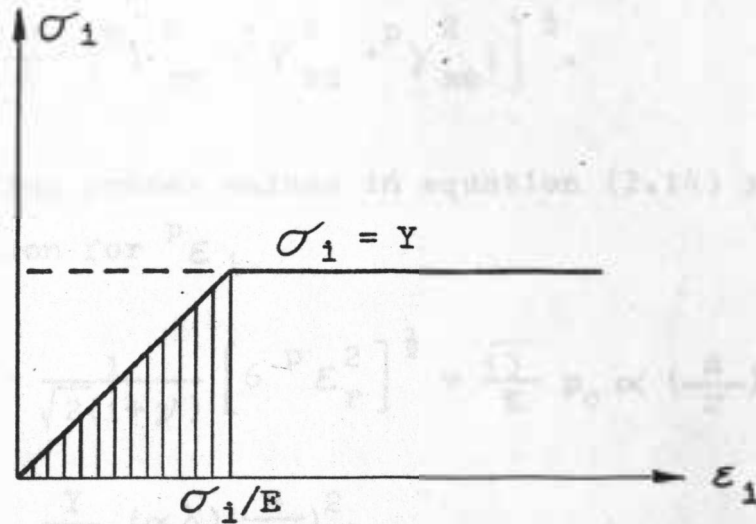


Figure 2.3

The term e_u denotes the density of energy in elastic region and for an incompressible body ($\nu = 1/2$) it simply equals $e_{\sigma_1}^2 / 2E$ which is the area under the curve shown in figure 2.3 in the elastic range. Similarly p_u denotes the density of energy in plastic region and it equals $Y p_{\epsilon_1} - Y^2 / 2E$. This is the area under the curve shown in figure 2.3 in its plastic range. Other notations are e_{σ_1} which is the stress intensity in elastic region and is given by equation (2.11a). Also, p_{ϵ_1} is the strain intensity in plastic region and is given as follows:

$$(2.14) \quad {}^p \epsilon_1 = \frac{1}{\sqrt{2(1+\nu)}} \left[({}^p \epsilon_r - {}^p \epsilon_z)^2 + ({}^p \epsilon_z - {}^p \epsilon_\theta)^2 + ({}^p \epsilon_\theta - {}^p \epsilon_r)^2 + \frac{3}{2} ({}^p \gamma_{r\theta}^2 + {}^p \gamma_{rz}^2 + {}^p \gamma_{z\theta}^2) \right]^{\frac{1}{2}}.$$

Substituting proper values in equation (2.14) yields a simple expression for ${}^p \epsilon_1$

$$(2.15) \quad {}^p \epsilon_1 = \frac{1}{\sqrt{2(1+\nu)}} \left[6 {}^p \epsilon_r^2 \right]^{\frac{1}{2}} = \frac{\sqrt{3}}{E} p_o \propto \left(\frac{a}{r} \right)^2 = \frac{Y}{E} (\propto \lambda) \left(\frac{a}{r} \right)^2.$$

The density of energy in the plastic region is

$$p_u = \frac{Y^2}{2E} \left[2 \propto \lambda \left(\frac{a}{r} \right)^2 - 1 \right],$$

while for the elastic region, the total strain energy is obtained by integration as

$$U_e = \int_{v_e} e_u dv = \int_{r^*}^{\infty} \int_0^{2\pi} \frac{3(p_o \propto)^2}{2E} \left(\frac{a}{r} \right)^4 d\theta dr$$

$$U_e = \frac{3\pi}{2E} (p_o \propto)^2 \frac{a^4}{r^{*2}}.$$

Substituting $\propto \lambda a^2$ for r^{*2} in the above equation gives

$$(2.16) \quad U_e = \int_{v_e} e_u dv = \frac{\pi}{2E} (\propto \lambda) Y^2 a^2$$

Similarly, the energy contained in the plastic region (part of which is stored and part is dissipated) is:

$$(2.17) \quad U_p = \int_{v_p} p_u dv = \frac{\gamma^2}{2E} \int_a^{r^*} \int_0^{2\pi} \left[2\alpha\lambda \left(\frac{a}{r} \right)^2 - 1 \right] d\theta r dr =$$

$$\frac{\pi \gamma^2 a^2}{2E} \left[4\alpha\lambda \ln \sqrt{\alpha\lambda} - \alpha\lambda + 1 \right].$$

To find the work done by the pressure p_0 on the displacement $u_r(a)$ we first assign (see figure 2.4)

$$T_1 = p_0 ; \quad ds = a d\theta$$

and obtain

$$\left[u_r(r) \right]_{r=a}^{r=a} = \frac{3}{2E} p_0 a \alpha.$$

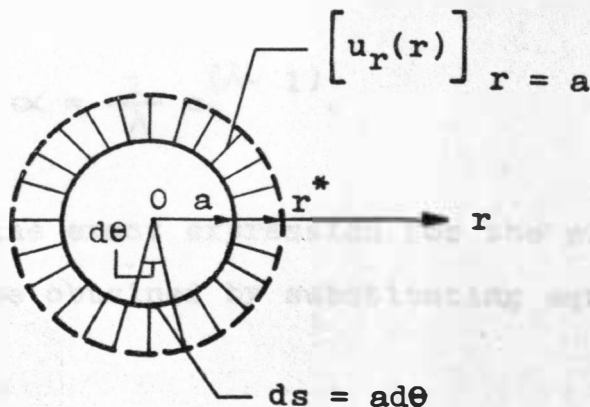


Figure 2.4

Thus

$$(2.18) \quad W = \int_S T_1 u_1 ds = \int_0^{2\pi} p_0 \left(-\frac{3}{2E} p_0 a \alpha \right) a d\theta =$$

$$\frac{\pi Y^2 a^2}{2E} 2\alpha \lambda^2 .$$

Now from equations (2.16), (2.17), and (2.18) the expression for potential energy of the system I^* can be found. Next, we require that I^* is minimum. This is done by differentiating the potential energy I^* with respect to α and equating the result to zero.

$$I^* = U - W = \frac{\pi Y^2 a^2}{2E} \left[4\alpha \lambda \ln(\sqrt{\alpha\lambda}) - 2\alpha \lambda^2 + 1 \right]$$

$$\frac{\partial I^*}{\partial \alpha} = \frac{\pi Y^2 a^2}{2E} \left[2\lambda (2 \ln(\sqrt{\alpha\lambda}) - \lambda + 1) \right] = 0$$

This expression solved for α gives:

$$(2.19) \quad \alpha = \frac{1}{\lambda} e^{(\lambda - 1)} .$$

From equation (2.12) the exact expression for the elastic-plastic boundary can be obtained by substituting equation (2.19) for α . Thus

$$(2.20) \quad r^* = a e^{(\lambda - 1)/2} .$$

Similarly, the stresses in the elastic region are obtained by combining equations (2.9) with (2.19). The result is

$$\begin{aligned}
 {}^e\sigma_r &= -\frac{Y}{\sqrt{3}} e^{\lambda-1} \left(\frac{a}{r}\right)^2 \\
 {}^e\sigma_\theta &= \frac{Y}{\sqrt{3}} e^{\lambda-1} \left(\frac{a}{r}\right)^2 \\
 {}^e\sigma_z &= 0
 \end{aligned}
 \tag{2.21}$$

It is interesting to note that when $\lambda \leq 1$ or $p_0 \leq Y/\sqrt{3}$; $r^* = a$ or, in other words, there is no plastic region. Therefore, the lower critical stress $p_0^* = Y/\sqrt{3}$ defines the value of applied load at which the plastic region begins to appear.

The equilibrium equation and the Huber-Mises-Hencky plasticity condition are used to derive the expressions for stresses in plastic region. Recalling that

$${}^p\sigma_1 = \frac{1}{\sqrt{2}} \left[({}^p\sigma_r - {}^p\sigma_\theta)^2 + ({}^p\sigma_\theta - {}^p\sigma_z)^2 + ({}^p\sigma_z - {}^p\sigma_r)^2 + 6({}^p\tau_{rz}^2 + {}^p\tau_{r\theta}^2 + {}^p\tau_{z\theta}^2) \right]^{\frac{1}{2}}$$

$${}^p\sigma_z = \frac{1}{2} ({}^p\sigma_\theta + {}^p\sigma_r)$$

$${}^p\tau_{rz} = {}^p\tau_{r\theta} = {}^p\tau_{z\theta} = 0$$

we have the stress intensity in the plastic region given as

$$^p\sigma_{\theta} = \frac{\sqrt{3}}{2} (^p\sigma_{\theta} - ^p\sigma_r) .$$

The governing equations for stress distribution inside the plastic region are:

$$(2.22a) \quad \frac{d \ ^p\sigma_r}{dr} + \frac{^p\sigma_r - ^p\sigma_{\theta}}{r} = 0$$

$$(2.22b) \quad ^p\sigma_{\theta} = Y = \frac{\sqrt{3}}{2} (^p\sigma_{\theta} - ^p\sigma_r) ;$$

with the following boundary conditions:

$$(2.23) \quad 1. \text{ at } r = a, \ ^p\sigma_r = -p_0$$

$$2. \text{ at } r = r^* \text{ where } r^* = a e^{(\lambda - 1)/2} ,$$

$$^p\sigma_r = ^e\sigma_r = -\frac{Y}{\sqrt{3}} .$$

From equation (2.22b) we have:

$$(2.24) \quad ^p\sigma_{\theta} = \frac{2Y}{\sqrt{3}} + ^p\sigma_r$$

Integrating the equation of equilibrium (2.22a) one gets an arbitrary constant. This constant is determined from the equation (2.23) and the expression for $^p\sigma_r$ which follows is an exact solution which is valid inside the plastic region:

$$(2.25) \quad {}^p\sigma_r = \frac{2Y}{\sqrt{3}} \ln\left(\frac{r}{a}\right) - p_0$$

similarly,

$$(2.26) \quad {}^p\sigma_\theta = \frac{2Y}{\sqrt{3}} \left[1 + \ln\left(\frac{r}{a}\right) \right] - p_0$$

and as expected, at $r = r^*$;

$${}^p\sigma_\theta = \frac{Y}{\sqrt{3}} = {}^e\sigma_\theta$$

Now we will show that the same results can be obtained for this case by the use of the variational principle (which usually gives approximate results).

To derive the equations (2.25) and (2.26), we will use the static equilibrium condition along with the Huber-Mises-Hencky plasticity condition (2.22b) and the result for α . The static condition will be satisfied if the sum of the internal forces in both elastic and plastic regions are equal to the resultant of the external forces applied to the body. Assuming unit thickness in Z direction, the static condition is:

$$(2.27) \quad \int_a^{r^*} {}^p\sigma_\theta dr + \int_{r^*}^\infty {}^e\sigma_\theta dr = a p_0$$

(see figure 2.5).

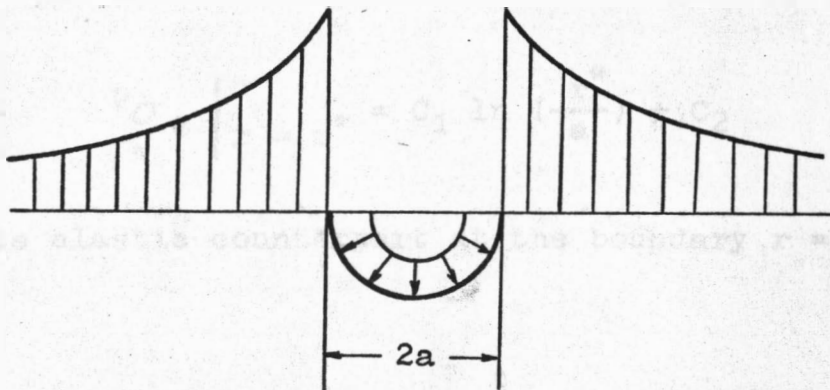


Figure 2.5

Now since ${}^e\sigma_\theta$ is known, the task will be to find ${}^p\sigma_\theta$ by resolving the integral equation (2.27). First we assume that the distribution of ${}^p\sigma_\theta$ has the form $C_1 \ln\left(-\frac{r}{a}\right) + C_2$; where C_1 and C_2 are arbitrary constants. With the aid of the boundary condition at $r = r^*$ and equation (2.27), the final form for ${}^p\sigma_\theta$ can be evaluated. Recalling that $\alpha\lambda = e^{\lambda-1}$ and $\sqrt{3} p_0 = Y\lambda$, the expression for ${}^e\sigma_\theta$ becomes

$$(2.28) \quad {}^e\sigma_{\theta} = p_0 \alpha \left(\frac{a}{r}\right)^2 = \frac{\alpha Y \lambda}{\sqrt{3}} \left(\frac{a}{r}\right)^2 ,$$

$$r^* = a \exp. \left(\frac{\lambda-1}{2}\right) .$$

By requiring that the stress

$$(2.29) \quad {}^p\sigma_{\theta} \Big|_{r=r^*} = C_1 \ln \left(\frac{r^*}{a}\right) + C_2$$

equals its elastic counterpart at the boundary $r = r^*$, we obtain

$$(2.29a) \quad {}^p\sigma_{\theta} = C_1 \frac{\lambda-1}{2} + C_2 = \frac{Y}{\sqrt{3}} .$$

Inserting equation (2.28) and expression $C_1 \ln \left(\frac{r}{a}\right) + C_2$ for ${}^p\sigma_{\theta}$ in the integral equation (2.27) enables us to integrate it. After some simplifications we have

$$(2.30) \quad C_1 \left[\frac{\lambda-1}{2} e^{\frac{\lambda-1}{2}} - e^{\frac{\lambda-1}{2}} + 1 \right] + C_2 \left[e^{\frac{\lambda-1}{2}} - 1 \right] + \frac{Y}{\sqrt{3}} e^{\frac{\lambda-1}{2}} = p_0 .$$

The system of equations (2.29a) and (2.30) can be solved simultaneously for constants C_1 and C_2 . The results are

$$(2.31) \quad C_1 = \frac{2Y}{\sqrt{3}} ; \quad C_2 = \frac{2Y}{\sqrt{3}} - p_0 .$$

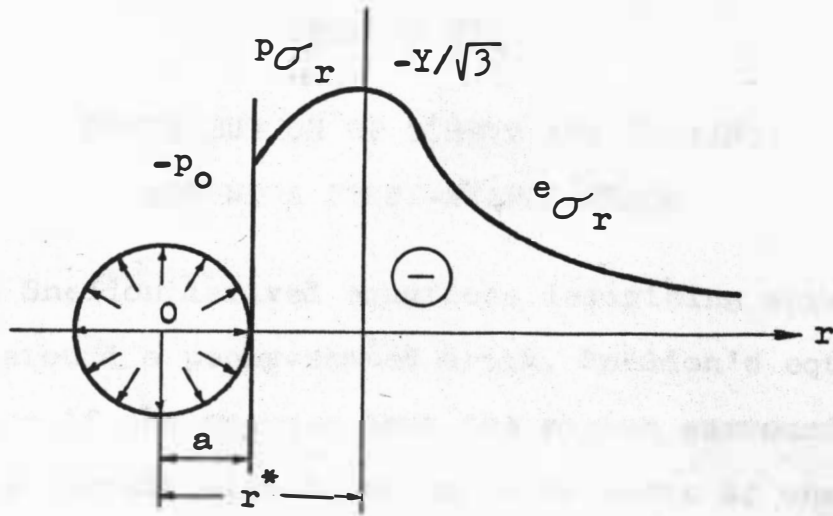
Putting values of C_1 and C_2 in expression $^p\sigma_\theta$ the best approximation solution for $^p\sigma_\theta$ is obtained. Then, $^p\sigma_r$ can be found easily from the equation (2.22b). It is noteworthy to observe that the solutions obtained for $^p\sigma_\theta$ and $^p\sigma_r$ by the approximation method are identical to the exact solutions given by the equations (2.25) and (2.26). The final results,

$$(2.32) \quad \sigma_\theta = \begin{cases} \frac{2Y}{\sqrt{3}} \left[1 + \ln \left(\frac{r}{a} \right) \right] - p_0 & a \leq r \leq r^* \\ \left[\frac{Y}{\sqrt{3}} \right] \left(\frac{a}{r} \right)^2 \exp. \left[\frac{\sqrt{3} p_0}{Y} - 1 \right] & r > r^* \end{cases}$$

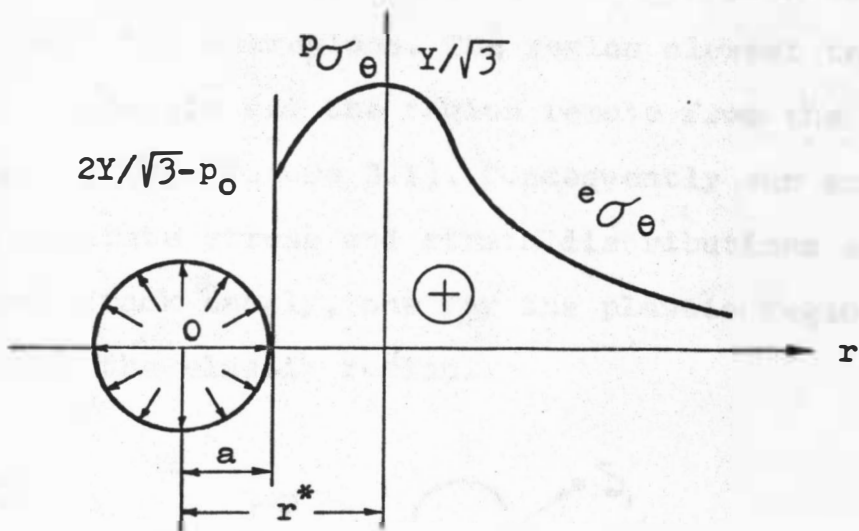
$$(2.33) \quad \sigma_r = \begin{cases} \frac{2Y}{\sqrt{3}} \ln \left(\frac{r}{a} \right) - p_0 & a \leq r \leq r^* \\ - \frac{Y}{\sqrt{3}} \left(\frac{a}{r} \right)^2 \exp. \left[\frac{\sqrt{3} p_0}{Y} - 1 \right] & r > r^* \end{cases}$$

$$r^* = a \exp. \left[\frac{\sqrt{3} p_0}{2Y} - \frac{1}{2} \right]$$

are illustrated in figure 2.6.



COMPRESSIVE STRESS



TENSILE STRESS

Figure 2.6

CHAPTER III

DISTRIBUTION OF STRESS AND STRAIN AROUND A PENNY-SHAPED CRACK

I. N. Sneddon derived equations describing stress distribution around a penny-shaped crack. Sneddon's equations are accurate if one assumes that the region surrounding the crack tip is purely elastic or in other words if one neglects the presence of the plastic region at the crack tip.

In this thesis, we have taken into account the presence of the plastic region. We have divided the region at the crack tip into two subregions. The region closest to the crack tip is plastic and the region remote from the crack tip is elastic (see figure 3.1). Consequently our analysis gives two separate stress and strain distributions around a penny-shaped crack namely, one for the plastic region and the other for the elastic region.

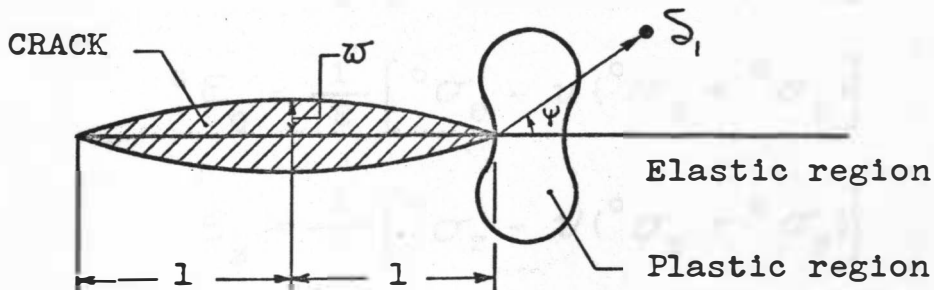


Figure 3.1 - Penny-shaped crack with elastic-plastic regions.

If p_0 denotes the stress applied perpendicular to the crack surface, ξ is the ratio of the coordinate ξ_1 to half of the major axis l , then according to Sneddon [16] the stress distribution around the crack tip in an elastic solid is as follows:

$$\begin{aligned}
 {}^0\sigma_r &= \frac{2 p_0}{\pi} (2 \xi)^{-\frac{1}{2}} \left[\frac{3}{4} \cos \frac{\psi}{2} + \frac{1}{4} \cos \frac{5\psi}{2} \right] + o(\xi) \\
 {}^0\sigma_\theta &= \frac{4 \nu p_0}{\pi} (2 \xi)^{-\frac{1}{2}} \cos \frac{\psi}{2} + o(\xi) \\
 (3.1) \quad {}^0\sigma_z &= \frac{2 p_0}{\pi} (2 \xi)^{-\frac{1}{2}} \left[\frac{5}{4} \cos \frac{\psi}{2} - \frac{1}{4} \cos \frac{5\psi}{2} \right] + o(\xi) \\
 {}^0\tau_{rz} &= \frac{p_0}{\pi} (2 \xi)^{-\frac{1}{2}} \sin \psi \cos \frac{3\psi}{2} + o(\xi) \\
 {}^0\tau_{r\theta} &= {}^0\tau_{z\theta} = 0 .
 \end{aligned}$$

Hooke's law applied in cylindrical coordinate system,

$$\begin{aligned}
 {}^0\varepsilon_r &= \frac{1}{E} \left[{}^0\sigma_r - \nu ({}^0\sigma_\theta + {}^0\sigma_z) \right] \\
 {}^0\varepsilon_\theta &= \frac{1}{E} \left[{}^0\sigma_\theta - \nu ({}^0\sigma_z + {}^0\sigma_r) \right] \\
 (3.2) \quad {}^0\varepsilon_z &= \frac{1}{E} \left[{}^0\sigma_z - \nu ({}^0\sigma_r + {}^0\sigma_\theta) \right] \\
 {}^0\gamma_{rz} &= \frac{2(1 + \nu)}{E} {}^0\tau_{rz} ;
 \end{aligned}$$

and equations (3.1) allow us to derive equations for purely elastic strains. The purely elastic strains are:

$$\begin{aligned}
 {}^0 \varepsilon_r &= \frac{p_0}{2\pi} \frac{1+\nu}{E} (2S)^{-\frac{1}{2}} \left[-2\sin\psi \sin \frac{3\psi}{2} + 4(1-2\nu) \cos \frac{\psi}{2} \right] \\
 &\quad + o(S) \\
 {}^0 \varepsilon_z &= \frac{p_0}{2\pi} \frac{1+\nu}{E} (2S)^{-\frac{1}{2}} \left[2\sin\psi \sin \frac{3\psi}{2} + 4(1-2\nu) \cos \frac{\psi}{2} \right] \\
 &\quad + o(S)
 \end{aligned}$$

(3.3)

$${}^0 \gamma_{rz} = \frac{2p_0}{\pi} \frac{1+\nu}{E} (2S)^{-\frac{1}{2}} \sin\psi \cos \frac{3\psi}{2} + o(S)$$

$${}^0 \varepsilon_\theta = 0$$

$${}^0 \gamma_{r\theta} = {}^0 \gamma_{z\theta} = 0$$

In order to obtain elastic and plastic strains, we make the same assumption as we made in chapter two; namely, we assume that the elastic and plastic strains are obtained from the purely elastic strains through multiplication of the purely elastic strains by an unknown parameter α which is to be found by the variational principle. We assume that,

$$(3.4) \quad {}^e \varepsilon_{ij} = {}^p \varepsilon_{ij} = \alpha {}^0 \varepsilon_{ij}.$$

Let us define $\lambda = p_0 / Y$, where Y is the yield stress of the material at uniaxial tension and thus λ denotes a dimensionless load. With this notation the elastic and plastic strains become:

$${}^e \varepsilon_r = {}^p \varepsilon_r = \frac{Y(\alpha\lambda)}{2\pi} \frac{1+\nu}{E} (2\delta)^{-\frac{1}{2}} \left[-2 \sin \Psi \sin \frac{3\Psi}{2} + 4(1-2\nu) \cos \frac{\Psi}{2} \right] + o(\delta)$$

$${}^e \varepsilon_z = {}^p \varepsilon_z = \frac{Y(\alpha\lambda)}{2\pi} \frac{1+\nu}{E} (2\delta)^{-\frac{1}{2}} \left[+2 \sin \Psi \sin \frac{3\Psi}{2} + 4(1-2\nu) \cos \frac{\Psi}{2} \right] + o(\delta)$$

$$(3.5) \quad {}^e \gamma_{rz} = {}^p \gamma_{rz} = \frac{Y(\alpha\lambda)}{2\pi} \frac{1+\nu}{E} (2\delta)^{-\frac{1}{2}} \left[2 \sin \Psi \cos \frac{3\Psi}{2} \right] + o(\delta)$$

$${}^e \varepsilon_\theta = {}^p \varepsilon_\theta = 0$$

$${}^e \gamma_{r\theta} = {}^p \gamma_{r\theta} = {}^e \gamma_{z\theta} = {}^p \gamma_{z\theta} = 0.$$

Knowing the elastic strains, the elastic stresses can be found by Hooke's law namely;

$${}^e\sigma_r = \alpha \circ \sigma_r = \frac{2Y(\alpha\lambda)}{\pi} (2\delta)^{-\frac{1}{2}} \left[\frac{3}{4} \cos \frac{\psi}{2} + \frac{1}{4} \cos \frac{5\psi}{2} \right] + o(\delta)$$

$${}^e\sigma_\theta = \alpha \circ \sigma_\theta = \frac{4\gamma Y(\alpha\lambda)}{\pi} (2\delta)^{-\frac{1}{2}} \cos \frac{\psi}{2} + o(\delta)$$

$$(3.6) \quad {}^e\sigma_z = \alpha \circ \sigma_z = \frac{2Y(\alpha\lambda)}{\pi} (2\delta)^{-\frac{1}{2}} \left[\frac{5}{4} \cos \frac{\psi}{2} - \frac{1}{4} \cos \frac{5\psi}{2} \right] + o(\delta)$$

$${}^e\tau_{rz} = \alpha \circ \tau_{rz} = \frac{Y(\alpha\lambda)}{\pi} (2\delta)^{-\frac{1}{2}} \left[\sin \psi \cos \frac{3\psi}{2} \right] + o(\delta)$$

$${}^e\tau_{r\theta} = {}^e\tau_{z\theta} = 0.$$

The plastic stresses cannot be found by Hooke's law since, Hooke's law does not apply in the plastic region. In order to get plastic stresses, we have made use of the Huber-Mises-Hencky plasticity condition and the Hencky-Ilyushin stress-strain law. In other words, we have applied the Huber-Mises-Hencky plasticity condition and the Hencky-Ilyushin law as constitutive equations in order to determine the stress distribution within the plastic region.

Now it is appropriate to restate the Huber-Mises-Hencky plasticity condition briefly: "Yielding takes place whenever the stress intensity (which is an invariant of the stress

tensor) reaches the yield stress of the material at simple tension ". This is a local criterion and it applies to each point of the body. The appropriate formula is given in extended form as follows:

$$(3.7) \quad {}^p\sigma_1 = \frac{1}{\sqrt{2}} \left[({}^p\sigma_r - {}^p\sigma_\theta)^2 + ({}^p\sigma_\theta - {}^p\sigma_z)^2 + ({}^p\sigma_z - {}^p\sigma_r)^2 \right. \\ \left. 6({}^p\tau_{r\theta}^2 + {}^p\tau_{rz}^2 + {}^p\tau_{z\theta}^2) \right]^{\frac{1}{2}} = Y.$$

The Hencky-Ilyushin law supplies a relationship between the increments of the plastic stresses and those of the plastic strains. The generalized modulus φ , which enters the Hencky-Ilyushin equations can be found from the Huber-Mises-Hencky plasticity condition. First let us apply the constitutive equations

$$(3.8a) \quad \alpha^1 \varepsilon = {}^p\varepsilon_r - {}^p\varepsilon_\theta = \varphi ({}^p\sigma_r - {}^p\sigma_\theta)$$

$$(3.8b) \quad \alpha^2 \varepsilon = {}^p\varepsilon_\theta - {}^p\varepsilon_z = \varphi ({}^p\sigma_\theta - {}^p\sigma_z)$$

$$(3.8c) \quad \alpha^3 \varepsilon = {}^p\varepsilon_z - {}^p\varepsilon_r = \varphi ({}^p\sigma_z - {}^p\sigma_r)$$

$$(3.8d) \quad \alpha^4 \gamma = {}^p\gamma_{rz} = 2\varphi {}^p\tau_{rz}.$$

As only three equations in (3.8) are independent, we need two more relationships in order to be able to solve for our five

unknowns, namely ${}^p\sigma_r$, ${}^p\sigma_\theta$, ${}^p\sigma_z$, ${}^p\tau_{rz}$, and φ . The fourth equation is the plasticity condition (3.7) and the last equation i.e. the dilatational strain is supplied by the theory of elasticity;*

$$(3.9) \quad {}^p\sigma_r + {}^p\sigma_\theta + {}^p\sigma_z = \frac{\alpha E}{1-2\nu} ({}^0\varepsilon_r + {}^0\varepsilon_\theta + {}^0\varepsilon_z) = F.$$

Rewriting the equation (3.8) in the form

$$(3.10a) \quad {}^p\sigma_r - {}^p\sigma_\theta = \frac{\alpha}{\varphi} {}^1\varepsilon$$

$$(3.10b) \quad {}^p\sigma_\theta - {}^p\sigma_z = \frac{\alpha}{\varphi} {}^2\varepsilon$$

$$(3.10c) \quad {}^p\sigma_z - {}^p\sigma_r = \frac{\alpha}{\varphi} {}^3\varepsilon$$

$$(3.10d) \quad {}^p\tau_{rz} = \frac{\alpha}{2\varphi} \gamma$$

where ${}^1\varepsilon$, ${}^2\varepsilon$, and ${}^3\varepsilon$ are known functions, defined by the equations (3.8) and (3.5); and substituting proper values from equations (3.10) in equation (3.7) we obtain an expression for the stress intensity in terms of the strain differences. Thus equation (3.7) becomes

* The experiments of Bridgeman [11] have demonstrated that even very high hydrostatic pressures did not affect the plastic yield. Therefore, the latter is assigned only to the deviatoric (shear) part of the stress tensor, while the normal stresses are assumed to produce only elastic type of deformation.

$$^p\sigma_1 = \frac{1}{\sqrt{2}} \left[\left(\frac{\alpha}{\varphi}\right)^2 {}^1\varepsilon^2 + \left(\frac{\alpha}{\varphi}\right)^2 {}^2\varepsilon^2 + \left(\frac{\alpha}{\varphi}\right)^2 {}^3\varepsilon^2 + 6\left(\frac{\alpha\gamma}{2\varphi}\right)^2 \right]^{\frac{1}{2}}.$$

According to Huber-Mises-Hencky plasticity condition which must be satisfied within the plastic region, we equate the stress intensity $^p\sigma_1$ to yield stress Y and solve for the plastic modulus φ :

$$Y = \frac{\alpha}{\sqrt{2}\varphi} \left[{}^1\varepsilon^2 + {}^2\varepsilon^2 + {}^3\varepsilon^2 + \frac{3}{2}\gamma^2 \right]^{\frac{1}{2}}$$

hence

$$(3.11) \quad \varphi = \frac{\alpha}{\sqrt{2}Y} \left[{}^1\varepsilon^2 + {}^2\varepsilon^2 + {}^3\varepsilon^2 + \frac{3}{2}\gamma^2 \right]^{\frac{1}{2}}.$$

After substitution of

$${}^1\varepsilon = ({}^p\varepsilon_r - {}^p\varepsilon_\theta) / \alpha$$

$${}^2\varepsilon = ({}^p\varepsilon_\theta - {}^p\varepsilon_z) / \alpha$$

$${}^3\varepsilon = ({}^p\varepsilon_z - {}^p\varepsilon_r) / \alpha$$

$$\gamma = {}^p\gamma_{rz} / \alpha.$$

in the expression for φ and some rearrangements, the equation (3.11) can be written in the following form

$$(3.12) \quad \varphi = \frac{1+\nu}{E} \frac{\alpha\lambda}{\pi} (2\delta)^{-\frac{1}{2}} A(\Psi) + o(\delta);$$

where

$$(3.13) \quad A(\Psi) = \left[3\sin^2 \Psi + 4(1-2\nu)^2 \cos^2 \frac{\Psi}{2} \right]^{\frac{1}{2}}.$$

On the other hand, the generalized modulus φ is related to the plastic strain intensity by the relationship

$$(3.14) \quad \varphi = \frac{1+\nu}{Y} {}^p \varepsilon_1$$

where ${}^p \varepsilon_1$ is defined as follows

$$(3.14a) \quad {}^p \varepsilon_1 = \frac{1}{\sqrt{2}(1+\nu)} \left[({}^p \varepsilon_r - {}^p \varepsilon_\theta)^2 + ({}^p \varepsilon_\theta - {}^p \varepsilon_z)^2 + ({}^p \varepsilon_z - {}^p \varepsilon_r)^2 + \frac{3}{2} ({}^p \gamma_{r\theta}^2 + {}^p \gamma_{rz}^2 + {}^p \gamma_{z\theta}^2) \right]^{\frac{1}{2}}.$$

Thus both φ and ${}^p \varepsilon_1$ are now determined through equations (3.12) and (3.14a). The remaining unknown functions ${}^p \sigma_r$, ${}^p \sigma_\theta$, and ${}^p \sigma_z$ can be determined from equations (3.10) and (3.9) which can be rewritten

$$p_{\sigma_r} - p_{\sigma_{\theta}} = {}^1\sigma$$

$$p_{\sigma_{\theta}} - p_{\sigma_z} = {}^2\sigma$$

(3.15)

$$p_{\sigma_z} - p_{\sigma_r} = {}^3\sigma$$

$$p_{\sigma_r} + p_{\sigma_{\theta}} + p_{\sigma_z} = F$$

where

$${}^1\sigma = \frac{Y \left[-2 \sin \psi \sin \frac{3\psi}{2} + 4(1-2\nu) \cos \frac{\psi}{2} \right]}{2 \left[3 \sin^2 \psi + 4(1-2\nu)^2 \cos^2 \frac{\psi}{2} \right]^{\frac{1}{2}}}$$

(3.16)

$${}^2\sigma = \frac{Y \left[-2 \sin \psi \sin \frac{3\psi}{2} - 4(1-2\nu) \cos \frac{\psi}{2} \right]}{2 \left[3 \sin^2 \psi + 4(1-2\nu)^2 \cos^2 \frac{\psi}{2} \right]^{\frac{1}{2}}}$$

$${}^3\sigma = \frac{Y \left[4 \sin \psi \sin \frac{3\psi}{2} \right]}{2 \left[3 \sin^2 \psi + 4(1-2\nu)^2 \cos^2 \frac{\psi}{2} \right]^{\frac{1}{2}}}$$

$$(3.16a) \quad F = 4 \frac{Y(\alpha\lambda)}{\pi} (1+\nu) (2\delta)^{-\frac{1}{2}} \cos \frac{\psi}{2} .$$

The expression for F has been obtained by inserting expressions (3.3) for the strains in equation (3.9).

The system of equations (3.16) can be easily solved for $P\sigma_r$, $P\sigma_\theta$, and $P\sigma_z$ in terms of $^1\sigma$, $^2\sigma$, φ , and F which are all known quantities. The results are:

$$\begin{aligned} P\sigma_r &= \frac{1}{3} (F + 2^1\sigma + ^2\sigma) \\ (3.17) \quad P\sigma_\theta &= \frac{1}{3} (F - ^1\sigma + ^2\sigma) \\ P\sigma_z &= \frac{1}{3} (F - ^1\sigma - 2^2\sigma); \end{aligned}$$

where $^1\sigma$ and $^2\sigma$ are given by the equations (3.16).

The final expressions for stresses within the plastic region are obtained by inserting equations (3.16) and (3.16a) in equations (3.17). After some simplifications the results become

$$(3.18) \quad P\sigma_r = \frac{1}{3} \left[M + \frac{Y}{A(\Psi)} (N-0) \right]$$

$$(3.19) \quad P\sigma_\theta = \frac{1}{3} \left[M - \frac{2Y}{A(\Psi)} N \right]$$

$$(3.20) \quad P\sigma_z = \frac{1}{3} \left[M + \frac{Y}{A(\Psi)} (N+0) \right]$$

where

$$M = \frac{4Y(\alpha\lambda)}{\pi} (1+\nu) (2\delta)^{-\frac{1}{2}} \cos \frac{\Psi}{2}$$

$$N = 2 (1 - 2\nu) \cos \frac{\Psi}{2}$$

$$0 = 3 \sin \Psi \sin \frac{3\Psi}{2} .$$

From equations (3.5), (3.8d), and (3.12) ${}^p\tau_{rz}$ becomes

$$(3.21) \quad {}^p\tau_{rz} = \frac{Y}{A(\Psi)} \sin \Psi \cos \frac{3\Psi}{2} .$$

Recalling that $\lambda = p_0 / Y$ and $A(\Psi)$ is given by the equation (3.13), the equations (3.18), (3.19), (3.20), and (3.21) describe the distribution of stresses in the plastic region. Strains in the plastic region are the same as strains in the elastic region and they are given by equations (3.5). Stresses in the elastic region are given by equations (3.6). For convenience, the final equations for the stress and strain distribution in the elastic and plastic regions are given below

$${}^e \varepsilon_r = {}^p \varepsilon_r = \frac{\alpha p_0}{2\pi} \frac{1+\nu}{E} (2\delta)^{-\frac{1}{2}} \left[-2 \sin \Psi \sin \frac{3\Psi}{2} + 4(1-2\nu) \cos \frac{\Psi}{2} \right] + o(\delta)$$

$${}^e \varepsilon_\theta = {}^p \varepsilon_\theta = 0$$

$$(3.22) \quad {}^e \varepsilon_z = {}^p \varepsilon_z = \frac{\alpha p_0}{2\pi} \frac{1+\nu}{E} (2\delta)^{-\frac{1}{2}} \left[2 \sin \psi \sin \frac{3\psi}{2} + 4(1-2\nu) \cos \frac{\psi}{2} \right] + o(\delta)$$

$${}^e \gamma_{rz} = {}^p \gamma_{rz} = \frac{\alpha p_0}{\pi} \frac{1+\nu}{E} (2\delta)^{-\frac{1}{2}} \left[2 \sin \psi \cos \frac{3\psi}{2} \right] + o(\delta)$$

$${}^e \gamma_{r\theta} = {}^p \gamma_{r\theta} = {}^e \gamma_{z\theta} = {}^p \gamma_{z\theta} = 0$$

$${}^e \sigma_r = \frac{2p_0\alpha}{\pi} (2\delta)^{-\frac{1}{2}} \left[\frac{3}{4} \cos \frac{\psi}{2} + \frac{1}{4} \cos \frac{5\psi}{2} \right] + o(\delta)$$

$${}^e \sigma_\theta = \frac{4\nu p_0\alpha}{\pi} (2\delta)^{-\frac{1}{2}} \cos \frac{\psi}{2} + o(\delta)$$

$$(3.23) \quad {}^e \sigma_z = \frac{2p_0\alpha}{\pi} (2\delta)^{-\frac{1}{2}} \left[\frac{5}{4} \cos \frac{\psi}{2} - \frac{1}{4} \cos \frac{5\psi}{2} \right] + o(\delta)$$

$${}^e \tau_{rz} = \frac{p_0\alpha}{\pi} (2\delta)^{-\frac{1}{2}} \left[\sin \psi \cos \frac{3\psi}{2} \right] + o(\delta)$$

$${}^e \tau_{r\theta} = {}^e \tau_{z\theta} = 0$$

$$p\sigma_r = \frac{1}{3} \left\{ \frac{p_0 \alpha}{\pi} (1+\nu)(2\delta)^{-\frac{1}{2}} \left[4 \cos \frac{\psi}{2} \right] + \frac{Y}{A(\psi)} \right. \\ \left. \left[-3 \sin \psi \sin \frac{3\psi}{2} + 2(1-2\nu) \cos \frac{\psi}{2} \right] \right\} + o(\delta)$$

$$p\sigma_\theta = \frac{1}{3} \left\{ \frac{p_0 \alpha}{\pi} (1+\nu)(2\delta)^{-\frac{1}{2}} \left[4 \cos \frac{\psi}{2} \right] - \frac{Y}{A(\psi)} \right. \\ \left. \left[4(1-2\nu) \cos \frac{\psi}{2} \right] \right\} + o(\delta)$$

$$(3.24) \quad p\sigma_z = \frac{1}{3} \left\{ \frac{p_0 \alpha}{\pi} (1+\nu)(2\delta)^{-\frac{1}{2}} \left[4 \cos \frac{\psi}{2} \right] + \frac{Y}{A(\psi)} \right. \\ \left. \left[3 \sin \psi \sin \frac{3\psi}{2} + 2(1-2\nu) \cos \frac{\psi}{2} \right] \right\} + o(\delta)$$

$$p\tau_{rz} = \frac{Y}{A(\psi)} \sin \psi \cos \frac{3\psi}{2} + o(\delta)$$

$$p\tau_{r\theta} = p\tau_{z\theta} = 0$$

where

$$A(\psi) = \left[3 \sin^2 \psi + 4(1-2\nu)^2 \cos^2 \frac{\psi}{2} \right]^{\frac{1}{2}}$$

and the unknown coefficient α is to be determined in the next chapter.

It should be noted that while the equations (3.24) hold inside the plastic region and the equations (3.23) are valid outside this region, the boundary between the two

regions has not yet been determined. However, this can be done if we recall that the elastic stress intensity

$$(3.25) \quad e\sigma_1 = \frac{Y(\alpha\lambda)}{\pi} (2\delta)^{-\frac{1}{2}} \left[3\sin^2 \Psi + 4(1-2\nu)^2 \cos^2 \frac{\Psi}{2} \right]^{\frac{1}{2}}$$

must satisfy the Huber-Mises-Hencky plasticity condition along the elastic-plastic boundary $\delta_* = \delta_*(\Psi)$. Equating (3.25) to Y and solving for δ_* we obtain

$$(3.26) \quad \delta_* = \frac{(\alpha\lambda)^2}{2\pi^2} \left[3\sin^2 \Psi + 4(1-2\nu)^2 \cos^2 \frac{\Psi}{2} \right].$$

Figure 1 illustrates the elastic-plastic boundary for values of the Poisson's ratio ν equal to 0.3 and 0.0. The angle Ψ_{\max} indicates the angle at which the radius δ_* is maximum.

It can be verified that along the boundary $\delta_* = \delta_*(\Psi)$ the stresses in elastic region are identical to the stresses in the plastic region and they are

$$(3.27) \quad \left[e\sigma_r \right]_{\delta=\delta_*} = \left[p\sigma_r \right]_{\delta=\delta_*} = \frac{2Y}{A(\Psi)} \left[\frac{3}{4} \cos \frac{\Psi}{2} + \frac{1}{4} \cos \frac{5\Psi}{2} \right]$$

$$(3.28) \quad \left[e\sigma_\theta \right]_{\delta=\delta_*} = \left[p\sigma_\theta \right]_{\delta=\delta_*} = \frac{4\nu Y}{A(\Psi)} \cos \frac{\Psi}{2}$$

$$(3.29) \quad \left[{}^e\sigma_z \right]_{\delta=\delta_*} = \left[{}^p\sigma_z \right]_{\delta=\delta_*} = \frac{2Y}{A(\Psi)} \left[-\frac{5}{4} \cos \frac{\Psi}{2} - \frac{1}{4} \cos \frac{5\Psi}{2} \right]$$

$$(3.30) \quad \left[{}^e\tau_{rz} \right]_{\delta=\delta_*} = \left[{}^p\tau_{rz} \right]_{\delta=\delta_*} = \frac{Y}{A(\Psi)} \sin \Psi \cos \frac{3\Psi}{2} ;$$

where the expression for $A(\Psi)$ is given by the equation (3.13).

Figure 2 illustrates the distribution of the normal stress σ_z and the stress intensity σ_1 in the elastic and plastic regions. For comparison the Sneddon's solution has been used to plot the distribution of the stresses in an elastic region.

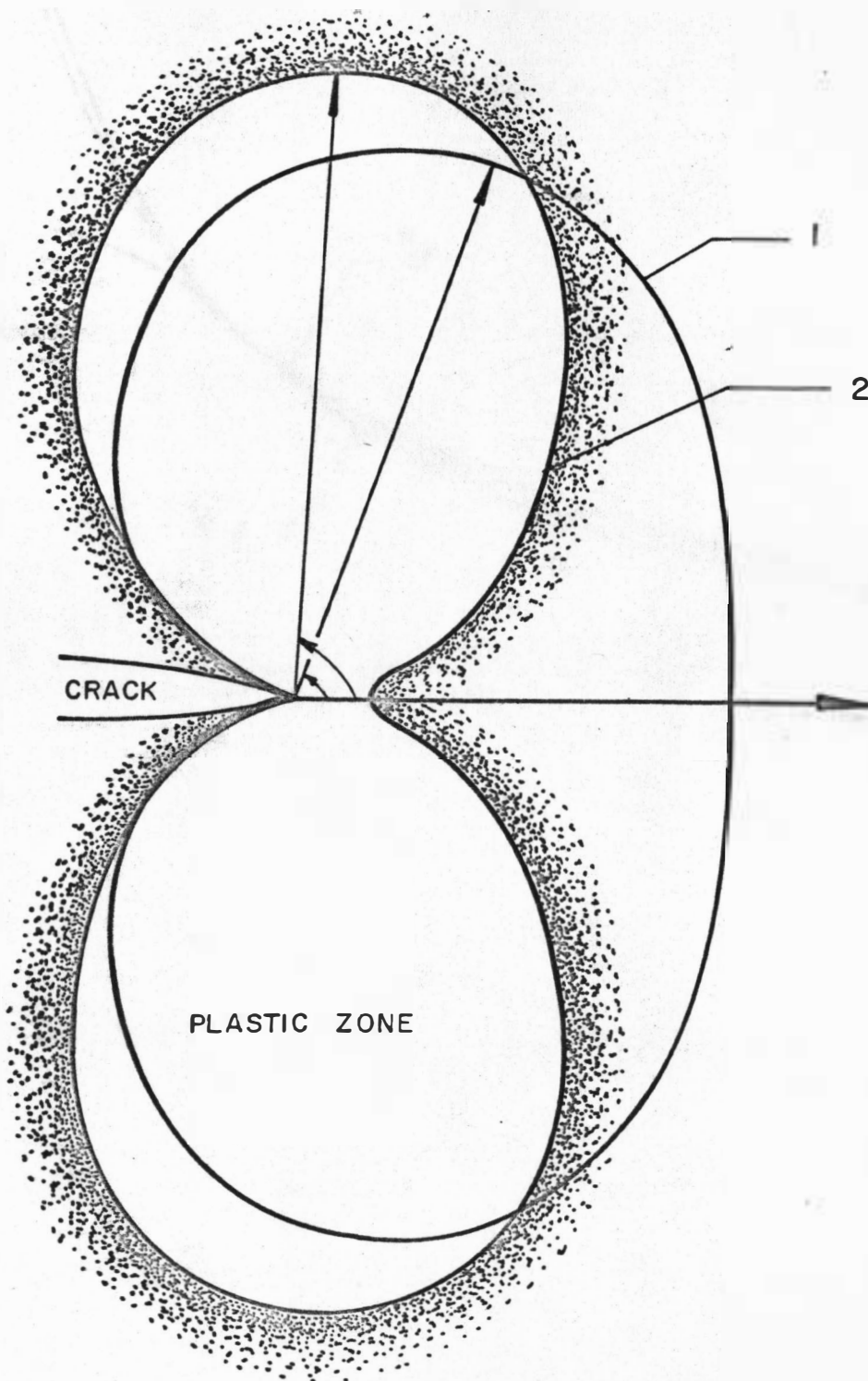


FIG.1 ELASTIC - PLASTIC BOUNDARY DERIVED FROM HUBER - MISES - HENCKY PLASTICITY CONDITION.

1. ($\nu = 0.0$; $\psi_{\max.} = 70^\circ$)
2. ($\nu = 0.3$; $\psi_{\max.} = 87^\circ$)

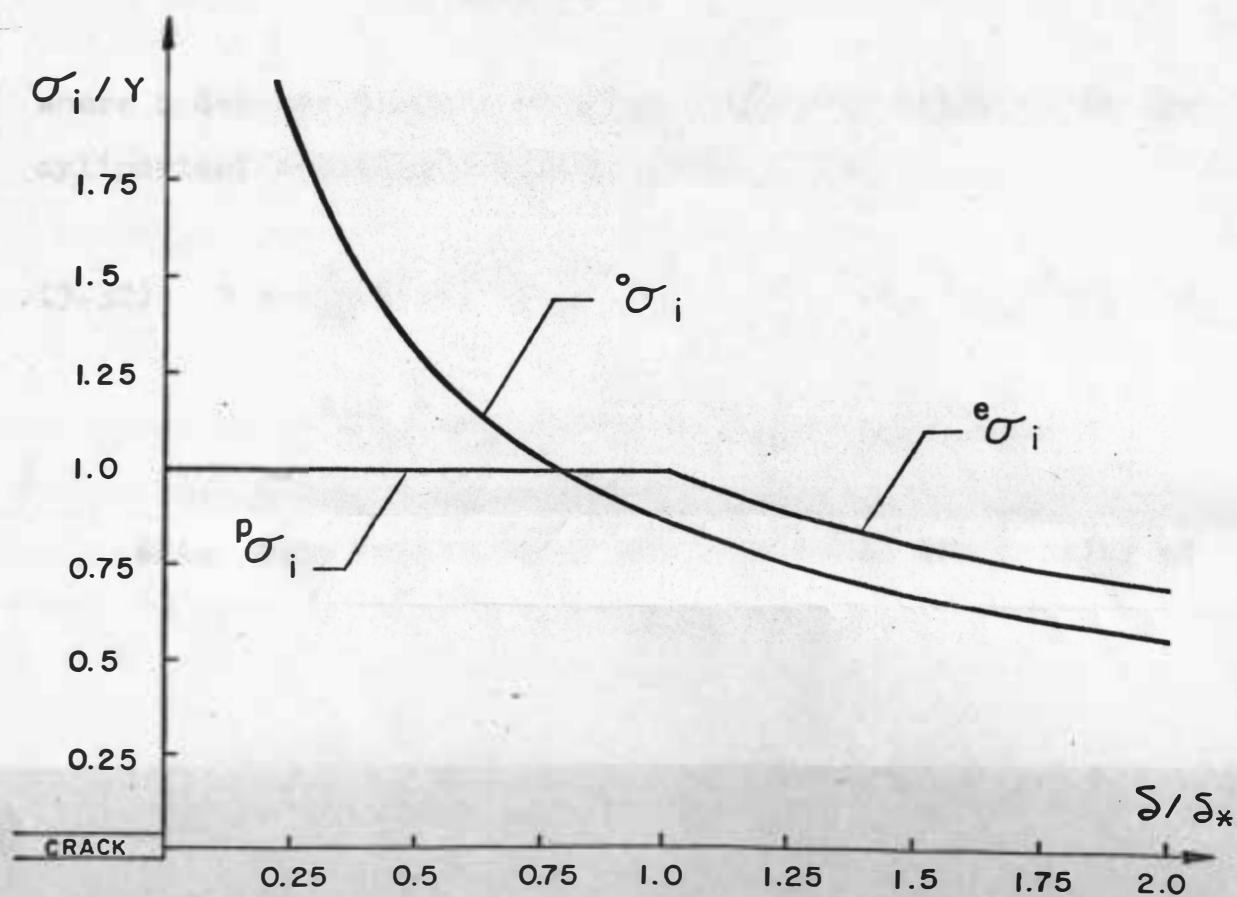
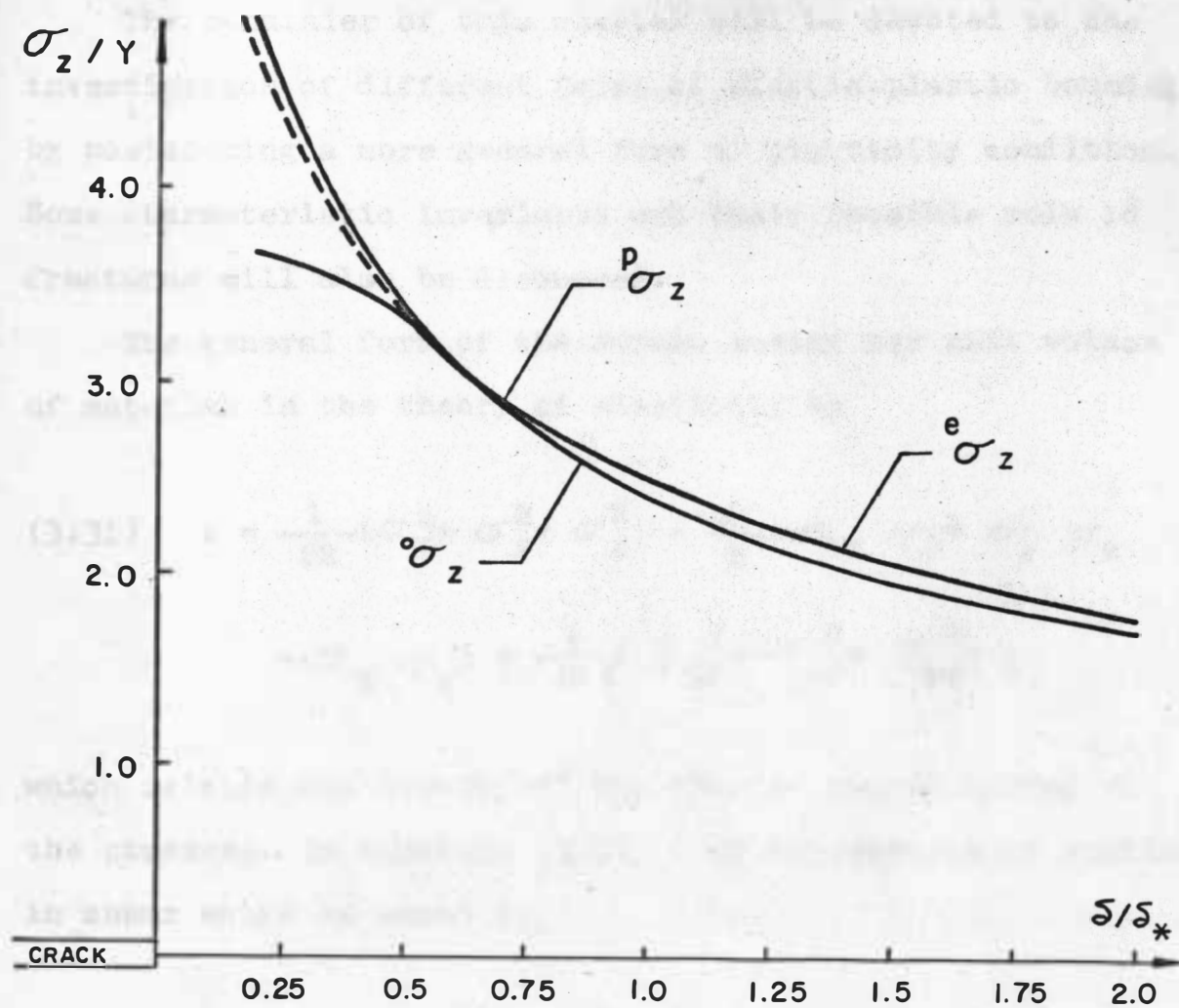


FIG. 2 DISTRIBUTION OF NORMAL STRESS σ_z AND STRESS INTENSITY σ_i
 $(\nu = 0.3 ; \psi = 0^\circ)$

The remainder of this chapter will be devoted to the investigation of different forms of elastic-plastic boundary by postulating a more general form of plasticity condition. Some characteristic invariants and their possible role in fractures will also be discussed.

The general form of the strain energy per unit volume of material in the theory of elasticity is

$$(3.31) \quad u = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\nu}{E}(\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_x \sigma_z) + \frac{1}{2G}(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2);$$

which relates the density of the elastic strain energy to the stresses. In equation (3.31) G is the modulus of rigidity in shear which is equal to

$$G = \frac{E}{2(1+\nu)}$$

where E denotes Young's modulus. The above equation in the cylindrical coordinate system reads

$$(3.32) \quad u = \frac{1}{2E}({}^e\sigma_r^2 + {}^e\sigma_\theta^2 + {}^e\sigma_z^2) - \frac{\nu}{E}({}^e\sigma_r {}^e\sigma_\theta + {}^e\sigma_\theta {}^e\sigma_z + {}^e\sigma_r {}^e\sigma_z) + \frac{1+\nu}{E}({}^e\tau_{r\theta}^2 + {}^e\tau_{rz}^2 + {}^e\tau_{\theta z}^2).$$

After some mathematical manipulations, the density of

the strain energy can be written as the sum of the volume energy and the distortion energy. The result is

$$(3.33) \quad u = \frac{3(1-2\nu)}{2E} \frac{1}{9} ({}^e\sigma_r + {}^e\sigma_\theta + {}^e\sigma_z)^2 + \frac{2}{3} (1+\nu) \frac{{}^e\sigma_1^2}{2E}$$

where the first term represents the density of the volume energy

$$u_v = \frac{3(1-2\nu)}{2E} \frac{1}{9} ({}^e\sigma_r + {}^e\sigma_\theta + {}^e\sigma_z)^2,$$

and the second term describes the density of the distortional energy

$$u_f = \frac{2}{3} (1+\nu) \frac{{}^e\sigma_1^2}{2E}.$$

Now, a general form of plasticity condition is postulated by assigning certain coefficients β and γ to the volume and distortion energies respectively, and equating the sum of these two terms to the critical density of the strain energy. Thus a more general equation for the plasticity condition is obtained

$$(3.34) \quad \beta u_v + \gamma u_f = u^*$$

where the critical strain energy

$$(3.35) \quad u^* = \frac{2}{3} (1+\nu) \frac{\gamma^2}{2E}$$

is chosen in a similar way as it was done by Huber.

Now let us put the proper values of elastic stresses in the equation (3.34) and solve it for δ_* in terms of the cofactors β and γ and the angle ψ . The elastic-plastic boundary becomes

$$(3.36) \quad \delta_* = \frac{1}{2} \left(\frac{\alpha\lambda}{\pi} \right)^2 \left\{ \beta \left[8(1+\nu)(1-2\nu) \cos^2 \frac{\psi}{2} \right] + \gamma \left[4(1-2\nu)^2 \cos^2 \frac{\psi}{2} + 3\sin^2 \psi \right] \right\}.$$

In figures 3a, b, c, and d the elastic-plastic boundary for $\nu = 0.3$ and different combinations of β and γ which respectively determine the roles of volume and distortion energies is graphed. Again ψ_{\max} indicates the angle at which δ_* attains the maximum value.

Note that when $\beta = 0.0$ and $\gamma = 1.0$ our yield condition reduces to the Huber-Mises-Hencky plasticity condition and the graph of the elastic-plastic boundary is identical to the graph shown in figure 1.

In order to determine the angle ψ at which the radius δ_* is maximum, we differentiate the equation (3.36) with respect to the independent variable ψ keeping the cofactors β and γ constant. The result for ψ_{\max} becomes

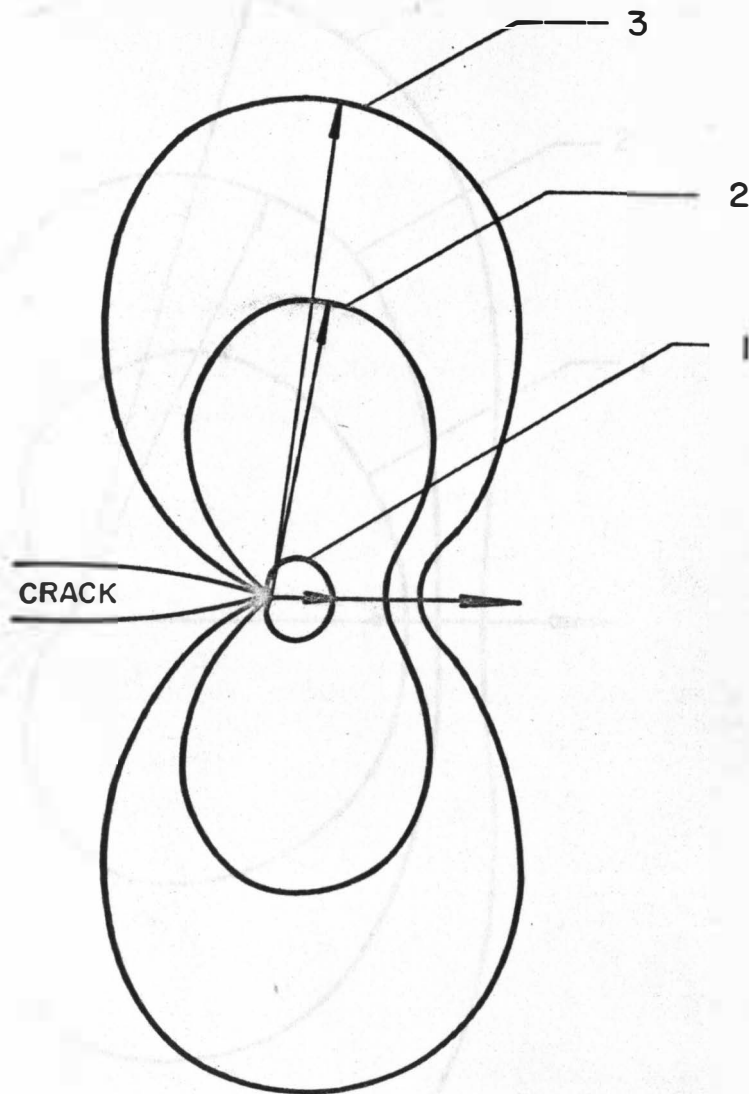


FIG.3a EFFECT OF YIELD CONDITION ON THE SHAPE OF PLASTIC ZONE AROUND THE PENNY SHAPED CRACK.

1. ($\beta = 0.2$; $\gamma = 0.073$; $\psi_{\max.} = 0^\circ$)
2. ($\beta = 0.2$; $\gamma = 0.600$; $\psi_{\max.} = 80^\circ$)
3. ($\beta = 0.2$; $\gamma = 1.000$; $\psi_{\max.} = 83^\circ$)

Decreasing value of γ corresponds to diminishing the role of the distortion energy.

Elastic-plastic boundary as derived from the generalized plasticity condition (β is a cofactor at the volumetric energy, γ is a cofactor at the distortional energy).

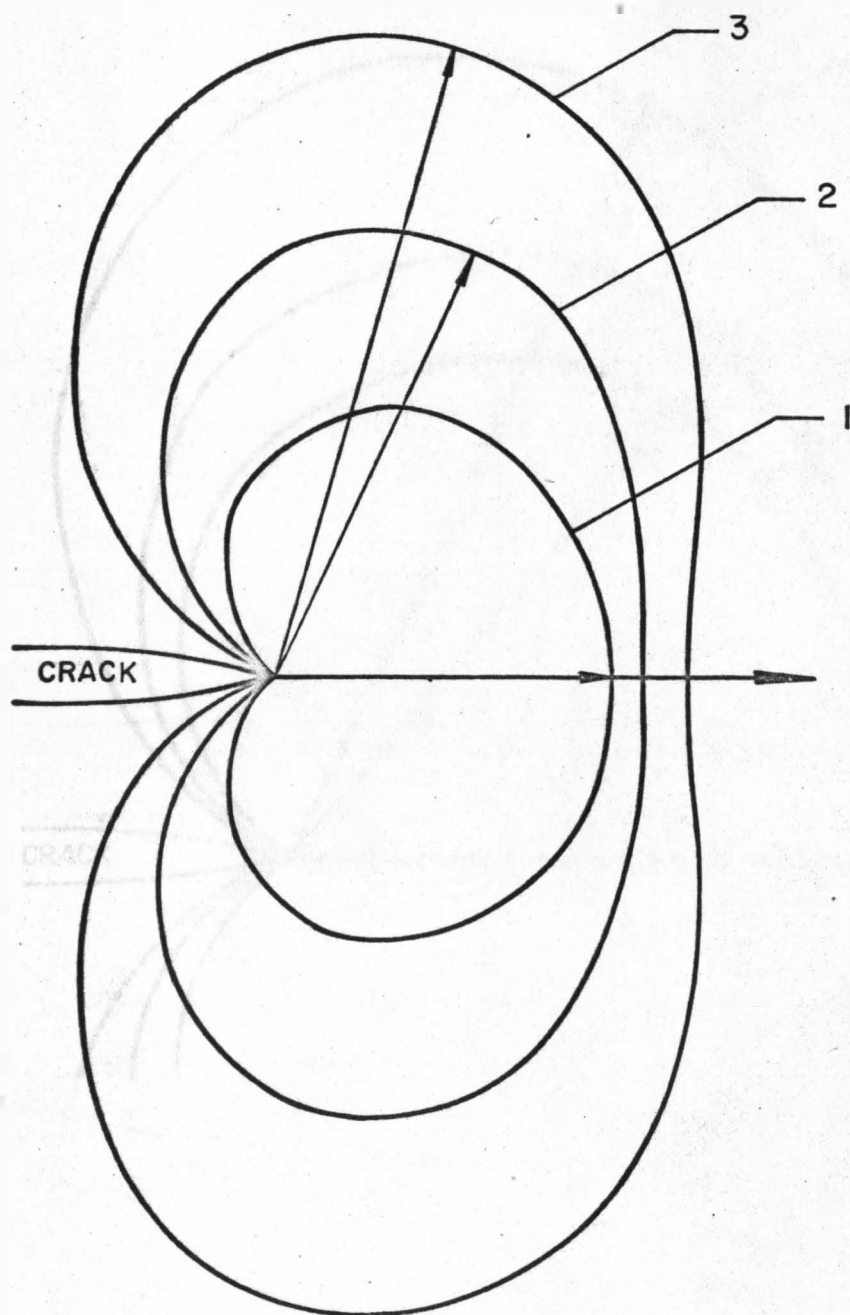
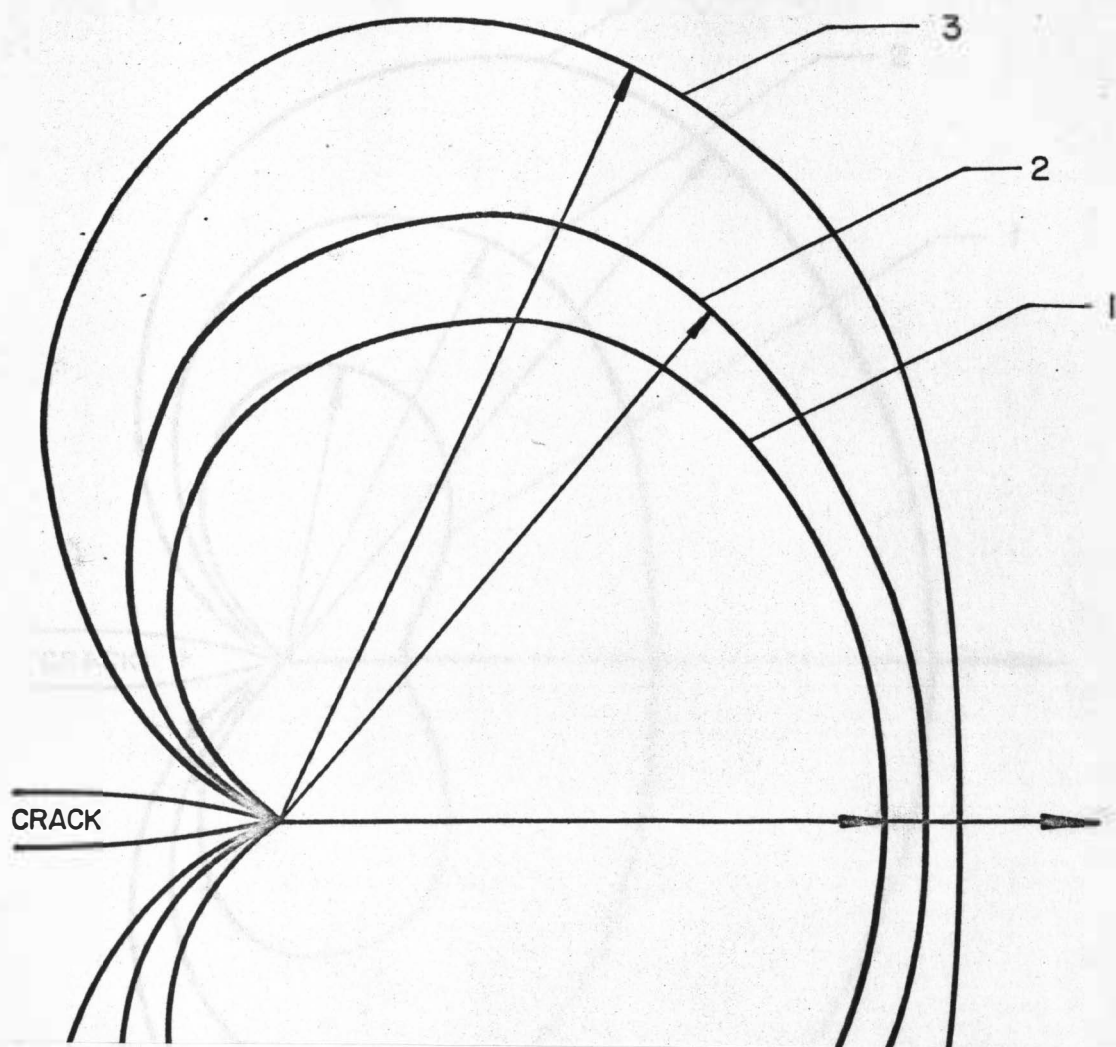


FIG.3b EFFECT OF YIELD CONDITION ON THE SHAPE OF PLASTIC ZONE AROUND THE PENNY SHAPED CRACK.

1. ($\beta = 0.6$; $\gamma = 0.220$; $\psi_{\max} = 0^\circ$)
2. ($\beta = 0.6$; $\gamma = 0.600$; $\psi_{\max} = 66^\circ$)
3. ($\beta = 0.6$; $\gamma = 1.000$; $\psi_{\max} = 75^\circ$)

Decreasing value of γ corresponds to diminishing the role of the distortion energy.

Elastic-plastic boundary as derived from the generalized plasticity condition (β is a cofactor at the volumetric energy, γ is a cofactor at the distortional energy).



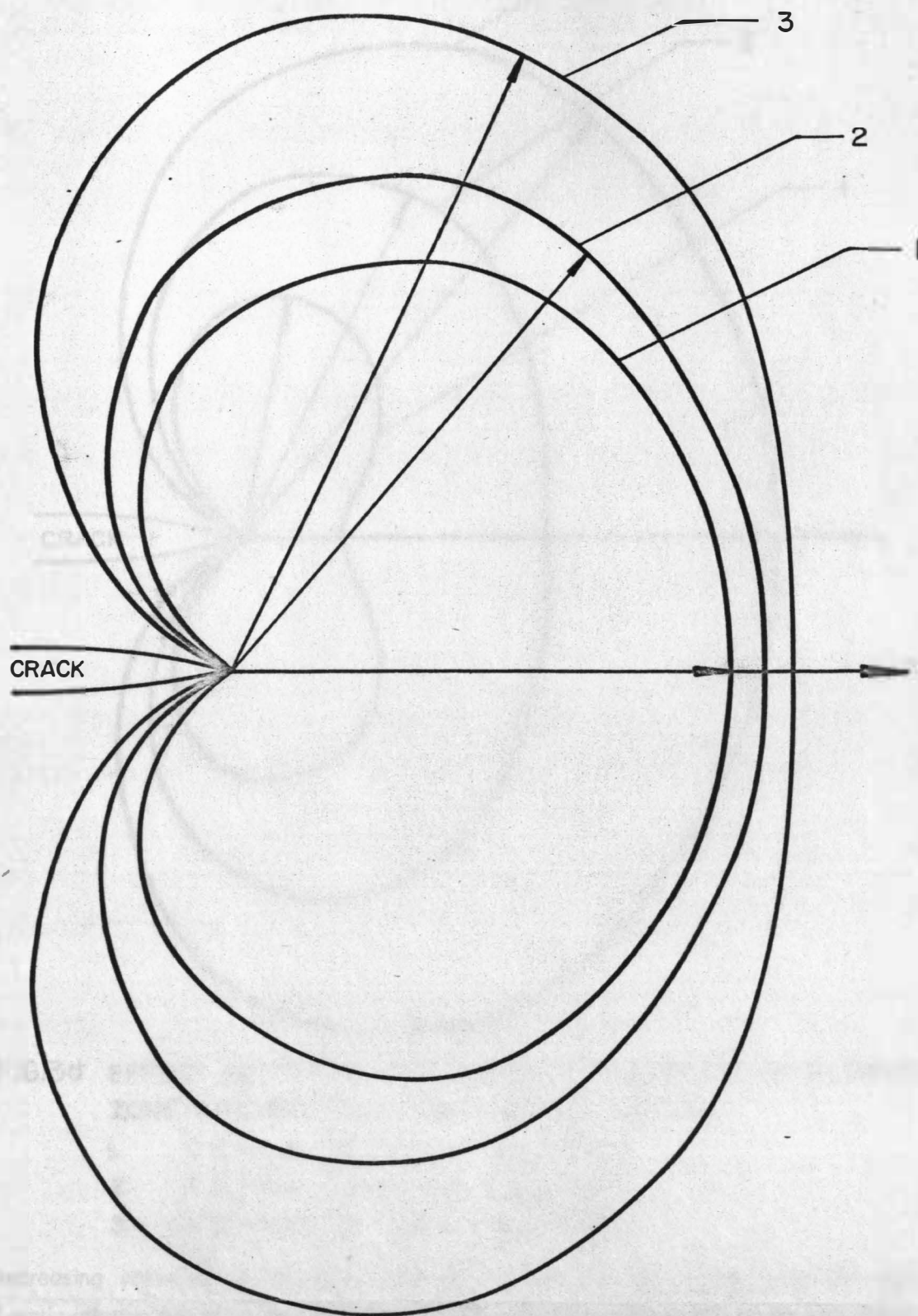


FIG.3c EFFECT OF YIELD CONDITION ON THE SHAPE OF PLASTIC ZONE AROUND THE PENNY SHAPED CRACK.

1. ($\beta = 1.0$; $\gamma = 0.366$; $\psi_{\max.} = 0^\circ$)
2. ($\beta = 1.0$; $\gamma = 0.600$; $\psi_{\max.} = 51^\circ$)
3. ($\beta = 1.0$; $\gamma = 1.000$; $\psi_{\max.} = 66^\circ$)

Decreasing value of γ corresponds to diminishing the role of the distortion energy.

Elastic-plastic boundary as derived from the generalized plasticity condition (β is a cofactor at the volumetric energy, γ is a cofactor at the distortional energy).

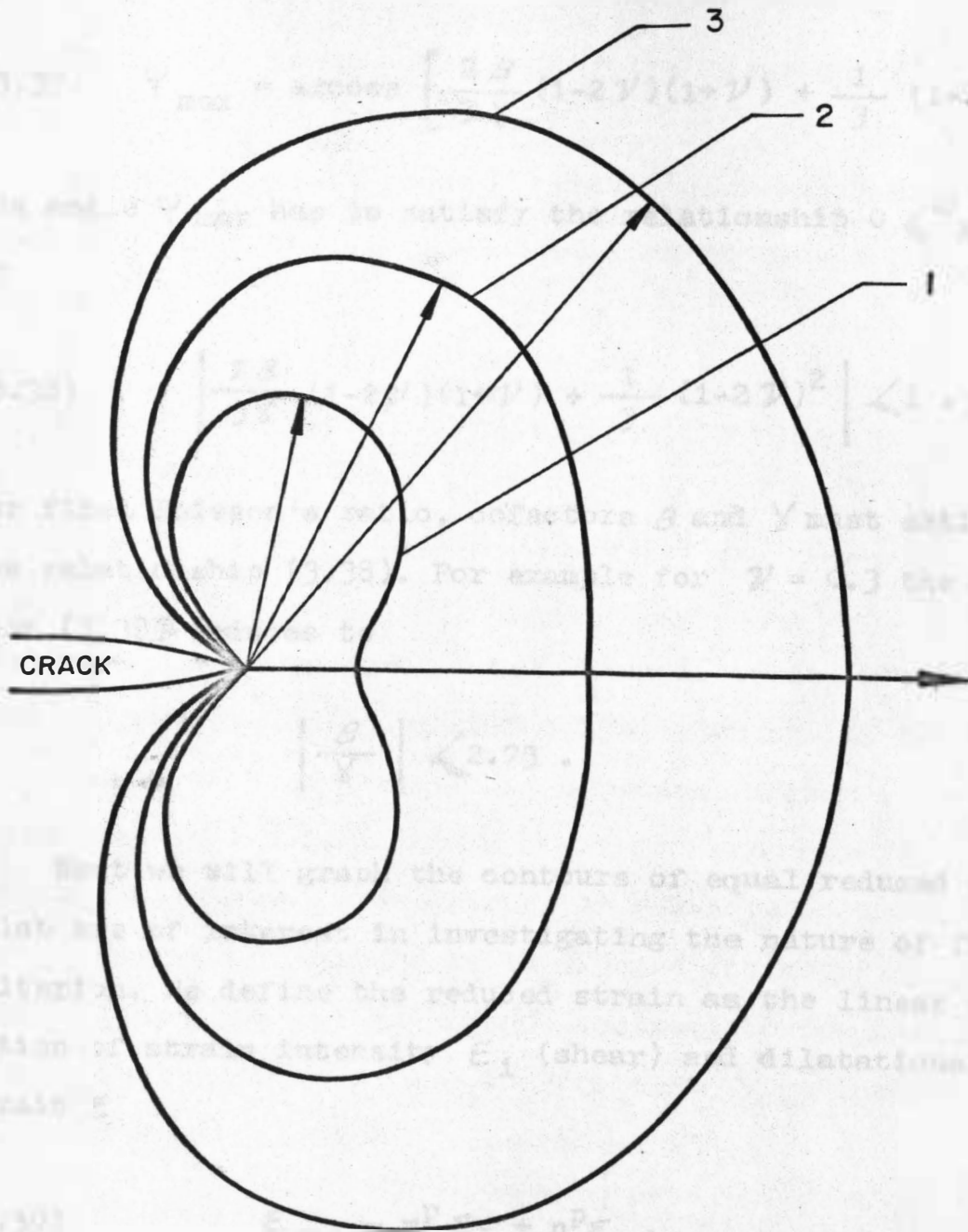


FIG.3d EFFECT OF YIELD CONDITION ON THE SHAPE OF PLASTIC ZONE AROUND THE PENNY SHAPED CRACK.

1. ($\gamma = 0.6$; $\beta = 0.2$; $\psi_{\max.} = 80^\circ$)
2. ($\gamma = 0.6$; $\beta = 0.6$; $\psi_{\max.} = 66^\circ$)
3. ($\gamma = 0.6$; $\beta = 1.0$; $\psi_{\max.} = 51^\circ$)

Decreasing value of β corresponds to diminishing the role of the volumetric energy.

Elastic-plastic boundary as derived from the generalized plasticity condition (β is a cofactor at the volumetric energy, γ is a cofactor at the distortional energy).

$$(3.37) \quad \Psi_{\max} = \arccos \left[\frac{2\beta}{3\gamma} (1-2\nu)(1+\nu) + \frac{1}{3} (1-2\nu)^2 \right].$$

The angle Ψ_{\max} has to satisfy the relationship $0 \leq \Psi_{\max} \leq \pi$ or

$$(3.38) \quad \left| \frac{2\beta}{3\gamma} (1-2\nu)(1+\nu) + \frac{1}{3} (1-2\nu)^2 \right| \leq 1.$$

For fixed Poisson's ratio, cofactors β and γ must satisfy the relationship (3.38). For example for $\nu = 0.3$ the equation (3.38) reduces to

$$\left| \frac{\beta}{\gamma} \right| \leq 2.73.$$

Next we will graph the contours of equal reduced strain which are of interest in investigating the nature of fracture criterion. We define the reduced strain as the linear combination of strain intensity ε_1 (shear) and dilatational strain ε

$$(3.39) \quad \varepsilon_{\text{red.}} = m^D \varepsilon_1 + n^D \varepsilon.$$

We require that $\varepsilon_{\text{red.}}$ be equal to a certain constant which we take as a multiple of strain at yield i.e. the ratio of the yield stress to the modulus of elasticity E . Next, the distortional strain (equal to strain intensity) and dilata-

tional strain are evaluated within the plastic region, while m and n are positive constants.

After all the necessary substitutions are made, the equation $\epsilon_{\text{red.}} = k \frac{Y}{E}$ is solved with respect to δ_* . The result

$$(3.40) \quad \delta_* = \frac{(\alpha\lambda)^2}{2\pi^2 k^2} \left\{ \left[4n(1-2\nu)(1+\nu) \cos \frac{\Psi}{2} \right] + m \left[3\sin^2 \Psi + 4(1-2\nu)^2 \cos^2 \frac{\Psi}{2} \right]^{\frac{1}{2}} \right\}^2$$

describes the contours of equal reduced strain, which is a three parameter family of curves. For each set of m , n , and k we have a different curve (at a chosen Poisson's ratio and a constant load λ).

Figure 4 shows the contours of equal reduced strain when the Poisson's ratio is equal to 0.3 and the coefficient k is equal to one. Several values are assigned to the coefficients m , and n as shown in figure 4. Here also Ψ_{max} indicates the angle at which radius δ_* is maximum.

FIG. 4

CONTOURS OF EQUAL REDUCED STRAIN
FOR $\nu = 0.3$ AND $k = 1$
WITH m AND n AS SHOWN

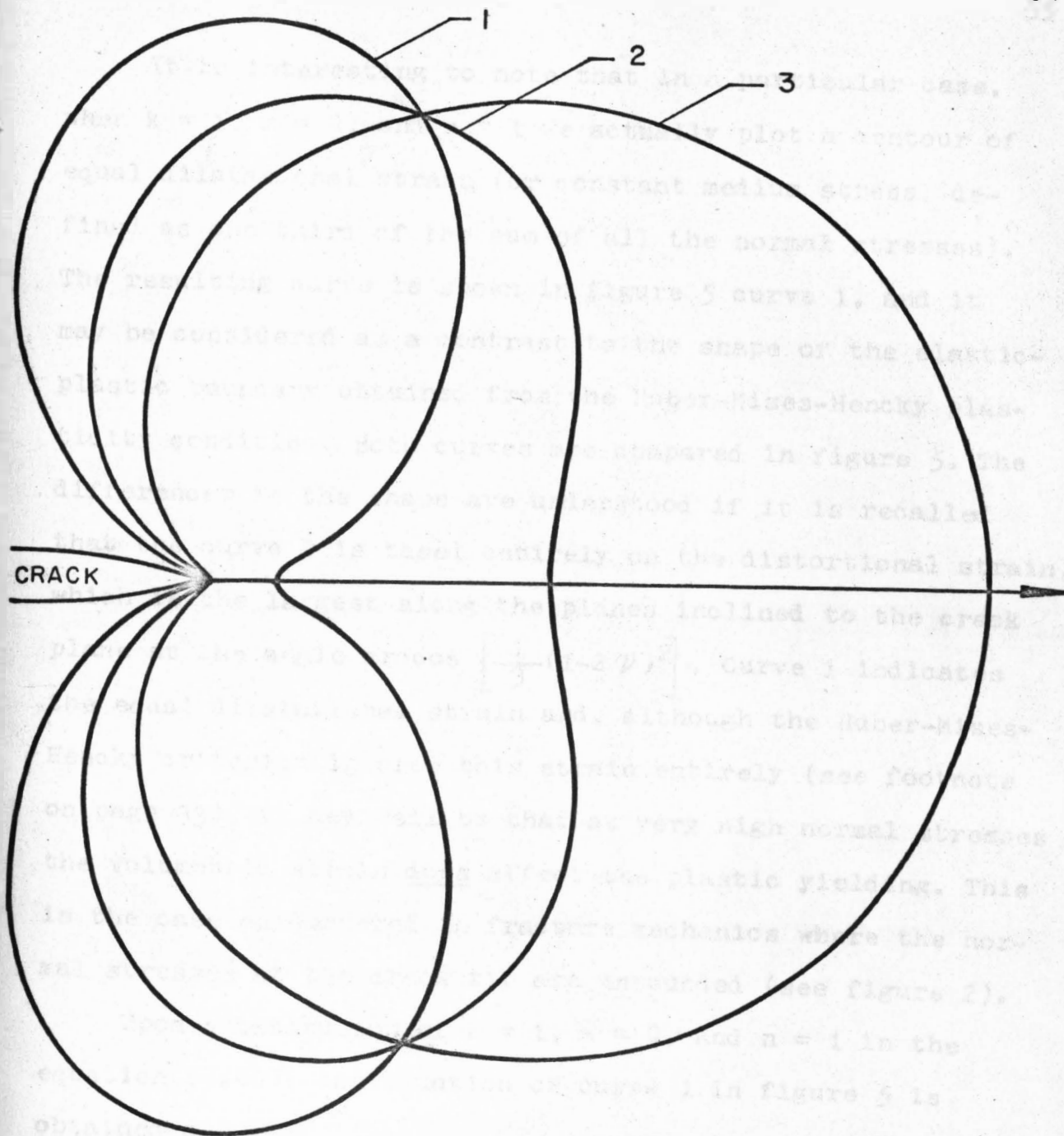


FIG. 4 CONTOURS OF EQUAL REDUCED STRAIN. THE REDUCED STRAIN CONSISTS OF DILATATIONAL STRAIN MULTIPLIED BY n , AND DISTORTIONAL STRAIN MULTIPLIED BY m .

1. ($m = 1$; $n = 0$; $\Psi_{\max.} = 87^\circ$)
2. ($m = 1/2$; $n = 1/2$; $\Psi_{\max.} = 80^\circ$)
3. ($m = 0$; $n = 1$; $\Psi_{\max.} = 0^\circ$)

It is interesting to note that in a particular case, when $k = 1$, $m = 0$, and $n = 1$ we actually plot a contour of equal dilatational strain (or constant medium stress; defined as one third of the sum of all the normal stresses). The resulting curve is shown in figure 5 curve 1, and it may be considered as a contrast to the shape of the elastic-plastic boundary obtained from the Huber-Mises-Hencky plasticity condition. Both curves are compared in figure 5. The differences in the shape are understood if it is recalled that the curve 2 is based entirely on the distortional strain, which is the largest along the planes inclined to the crack plane at the angle $\arccos \left[-\frac{1}{3}(1-2\nu)^2 \right]$. Curve 1 indicates the equal dilatational strain and, although the Huber-Mises-Hencky criterion ignores this strain entirely (see footnote on page 33), it may well be that at very high normal stresses the volumetric strain does affect the plastic yielding. This is the case encountered in fracture mechanics where the normal stresses at the crack tip are unbounded (see figure 2).

Upon substitution of $k = 1$, $m = 0$, and $n = 1$ in the equation (3.40); the equation of curve 1 in figure 5 is obtained as

$$(3.41) \quad \mathcal{J}_* = \frac{8(\alpha\lambda)^2}{\pi^2} \left[(1+\nu)(1-2\nu) \cos \frac{\psi}{2} \right]^2 .$$

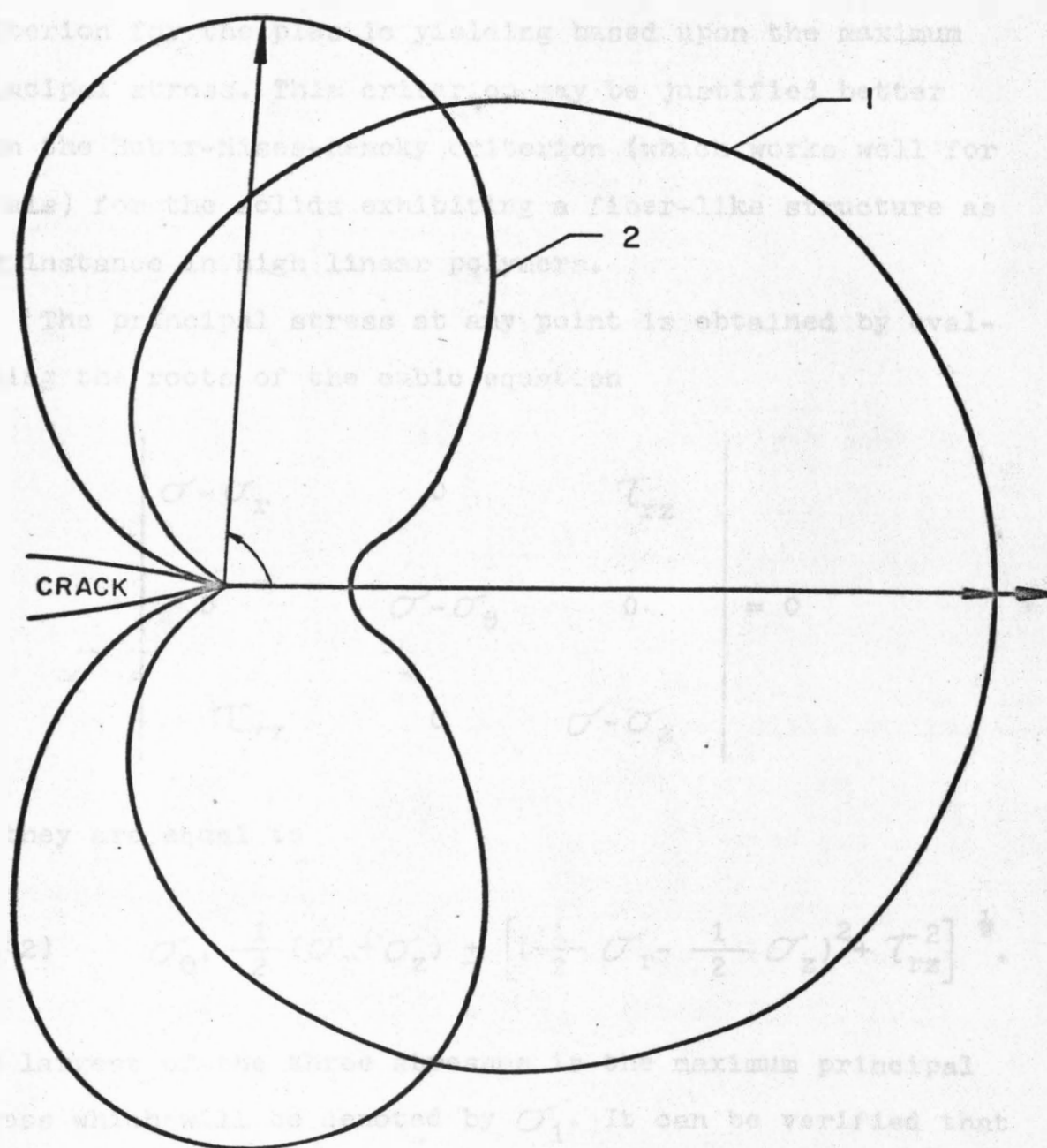


FIG. 5 CONTOURS OF EQUAL DILATATIONAL STRAIN
AND EQUAL DISTORTIONAL STRAIN.

1. Equal dilatational strain ($\nu = 0.3$; $\Psi_{\max.} = 0^\circ$)
2. Equal distortional strain ($\nu = 0.3$; $\Psi_{\max.} = 87^\circ$)

To conclude this chapter, we shall briefly discuss the criterion for the plastic yielding based upon the maximum principal stress. This criterion may be justified better than the Huber-Mises-Hencky criterion (which works well for metals) for the solids exhibiting a fiber-like structure as for instance in high linear polymers.

The principal stress at any point is obtained by evaluating the roots of the cubic equation

$$\begin{vmatrix} \sigma - \sigma_r & 0 & \tau_{rz} \\ 0 & \sigma - \sigma_\theta & 0 \\ \tau_{rz} & 0 & \sigma - \sigma_z \end{vmatrix} = 0$$

so they are equal to

$$(3.42) \quad \sigma_\theta, \frac{1}{2}(\sigma_r + \sigma_z) \pm \left[\left(\frac{1}{2} \sigma_r - \frac{1}{2} \sigma_z \right)^2 + \tau_{rz}^2 \right]^{\frac{1}{2}}.$$

The largest of the three stresses is the maximum principal stress which will be denoted by σ_1 . It can be verified that

$$(3.43) \quad \sigma_1 = \frac{1}{2}(\sigma_r + \sigma_z) + \left[\left(\frac{1}{2} \sigma_r - \frac{1}{2} \sigma_z \right)^2 + \tau_{rz}^2 \right]^{\frac{1}{2}}.$$

To obtain the lines of equal maximum principal stress we equate the maximum principal stress σ_1 to a constant

taken as a multiple of the applied load p_0 . Inserting proper values of stresses in the elastic region from the equations (3.25) in the equation (3.43) and after some simplifications the expression for maximum principal stress in the elastic region becomes

$$(3.44) \quad e\sigma_1 = \frac{p_0 \alpha}{\pi} (2\delta)^{-\frac{1}{2}} \left[2 \cos \frac{\psi}{2} + \sin \psi \right] + o(\delta).$$

Equating the above expression to $k p_0$ where k is a positive coefficient and solving for δ_* we obtain

$$\delta_* = \frac{1}{2} \left(\frac{\alpha}{k\pi} \right)^2 \left[2 \cos \frac{\psi}{2} + \sin \psi \right]^2.$$

Figure 6 shows three graphs of lines of equal maximum principal stress in the elastic region by assigning values of 2.0, 1.0, and 0.9 to k while, ψ_{\max} indicates the angle at which δ_* is maximum.

Now inserting proper values of stresses in the plastic region from equations (3.26) in the equation (3.43) and simplifying the expression for maximum principal stress in the plastic region we get

$$(3.45) \quad p\sigma_1 = \frac{4}{3} \frac{p_0 \alpha}{\pi} (2\delta)^{-\frac{1}{2}} (1+\nu) \cos \frac{\psi}{2} + \frac{Y}{A(\psi)} \\ \left[-\frac{2}{3} (1-2\nu) \cos \frac{\psi}{2} + \sin \psi \right] + o(\delta)$$

where $A(\psi)$ is given by equation (3.13). Similarly equating the above expression to kY where k is a positive coefficient and solving for δ_* gives

$$\delta_* = \frac{1}{2} \left[\frac{4 P_0 \alpha}{3 \pi Y} (1 + \nu) \right]^2 \left[\frac{k}{\cos \frac{\psi}{2}} - \frac{1}{A(\psi)} \right. \\ \left. \left(\frac{2}{3} (1 - 2\nu) + 2 \sin \frac{\psi}{2} \right) \right]^{-2}.$$

Figure 7 shows graphs of lines of equal maximum principal stress evaluated within the plastic region when Poisson ratio is equal to 0.3 and k is given values of 2.0, 1.0, and 0.9. The curves which are shown by dotted lines are drawn by assigning the values 0.3, 0.2, and 0.1 to k . In this case the formulae for plastic stresses do not apply because the appropriate contours become so large that they extend beyond the plastic region and are partly located in the elastic region.* Consequently, the complete curves are not drawn and only some sections are shown, but it is interesting to note that the region encompassed by the $P\sigma_1 = \text{constant}$ line becomes narrow and very long for low values of k . The maximum for curves with solid lines occurs at angles between 30° and

* The solution presented here is only asymptotic i.e., it is valid in the region close to the crack tip where the stresses are large. For smaller values of k we depict the points of lower principal stress, so the line $\sigma_1 = \text{constant}$ escapes beyond the limits of applicability of the present theory.

40°, while the maximum of the dotted curves is at angles very close to zero. This pattern seems to approach the shape of the plastic zone assumed in the Barenblatt-Dugdale model of the crack [1].



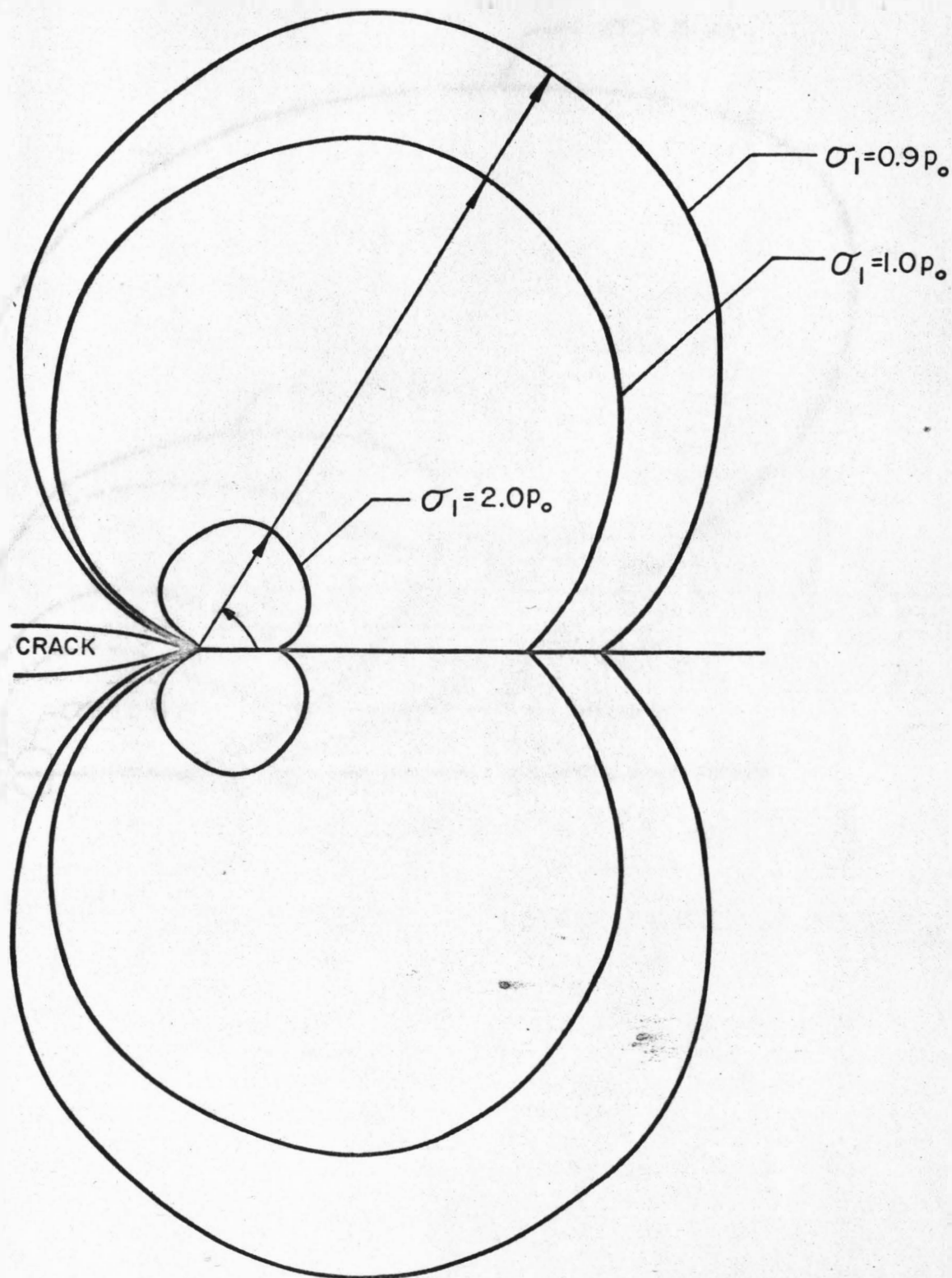


FIG.6 LINES OF EQUAL MAXIMUM PRINCIPAL STRESS EVALUATED
WITHIN THE ELASTIC REGION.

$$\Psi_{\max.} = 60^\circ (\text{ALL CASES})$$

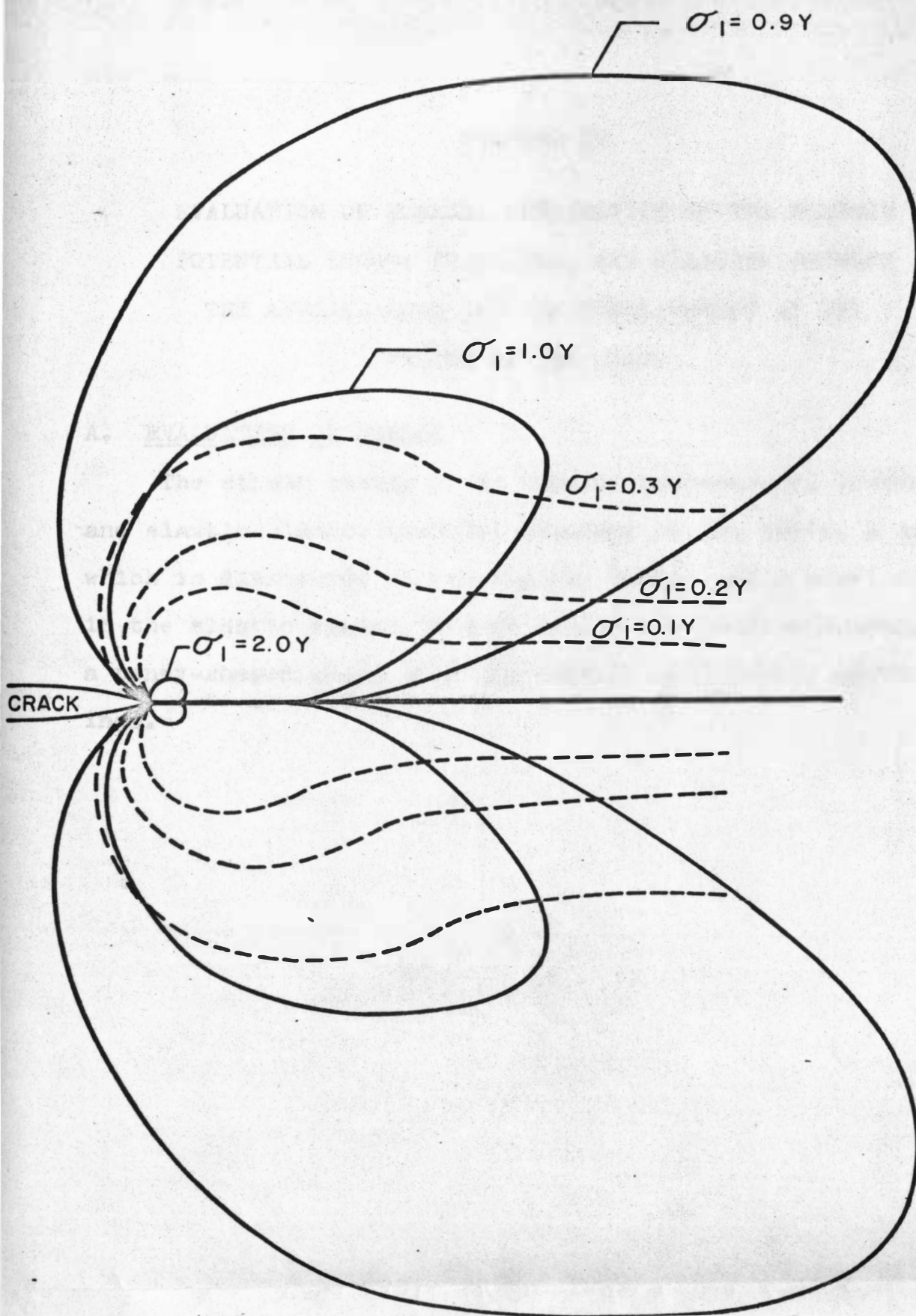


FIG.7 LINES OF EQUAL MAXIMUM PRINCIPAL STRESS EVALUATED WITHIN THE PLASTIC REGION.

$$\nu = 0.3 \text{ (ALL CASES)}$$

Dotted lines indicate that the region of applicability of the present solution is surpassed.

CHAPTER IV

EVALUATION OF ENERGY, APPLICATION OF THE MINIMUM POTENTIAL ENERGY PRINCIPLE, AND RELATION BETWEEN THE APPLIED LOAD AND THE DISPLACEMENT AT THE CENTER OF THE CRACK

A. EVALUATION OF ENERGY

The strain energy of an ideally homogeneous, isotropic, and elastic-plastic material consists of two parts. A part which is dissipated in the plastic region and a part stored in the elastic region. Figure 4.1 is a schematic drawing of a penny-shaped crack with its plastic and elastic surroundings.

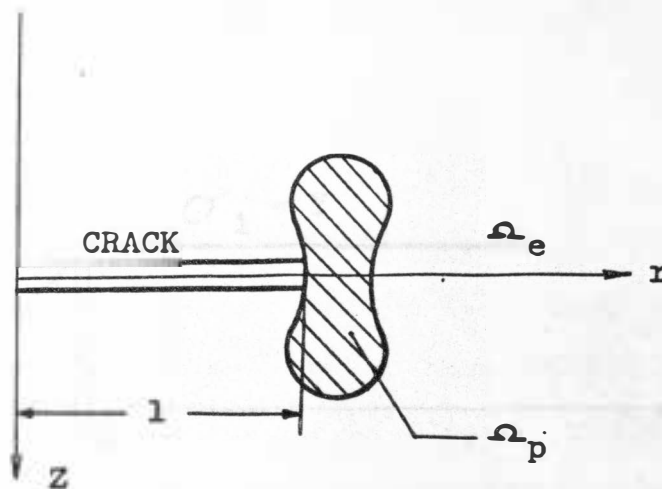


Figure 4.1

The total strain energy in terms of energy densities in the elastic and plastic regions is given by the following integrals

$$(4.1) \quad U = W_e + W_p = \int_{\Omega_p} p_u \, dv + \int_{\Omega_e} e_u \, dv$$

where p_u and e_u are densities of energy in the plastic and elastic regions respectively. Furthermore, the strain energy in each region has two components namely, distortion energy and volume energy. The distortional strain energy is proportional to the area under the curve of the stress-strain relation (see figure 4.2) while, the volume strain energy is given in both regions by the relationship known from the theory of elasticity $u_v = \frac{3(1-2\nu)}{2E} \sigma_m^2$, where σ_m denotes the medium stress.

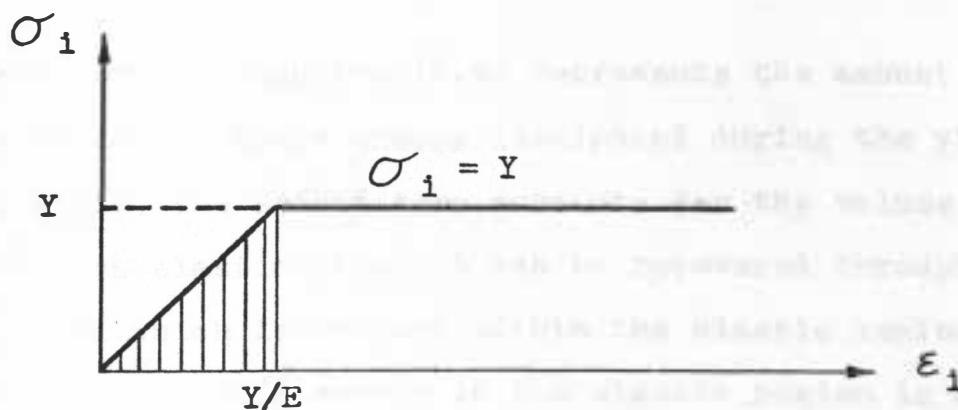


Figure 4.2

Thus, the elastic strain energy density in terms of the distortion and volume energies is written

$$(4.2) \quad e_u = \frac{2}{3}(1+\nu) \frac{e\sigma_1^2}{2E} + \frac{3(1-2\nu)}{2E} \sigma_m^2.$$

Here $e\sigma_1$ denotes the stress intensity within the elastic region and it was defined in the previous chapter while, $e\sigma_m$ is the medium stress defined as follows

$$(4.3) \quad e\sigma_m = \frac{e\sigma_r + e\sigma_\theta + e\sigma_z}{3}$$

The strain energy density for the plastic region can also be written in terms of the distortion and volume energies namely,

$$(4.4) \quad p_u = \frac{2}{3}(1+\nu) \left[Y^p \varepsilon_1 - \frac{Y^2}{2E} \right] + \frac{3(1-2\nu)}{2E} p\sigma_m^2.$$

The first term in equation (4.4) represents the amount of the distortional strain energy dissipated during the yielding process while, the second term accounts for the volume energy which remains elastic (i.e. it can be recovered through unloading) and which is present within the plastic region.

The total strain energy in the elastic region is obtained by integrating the strain energy density over the volume of the elastic region. The volume of the elastic region is the difference between the total volume of material

and the volume of the plastic region. Hence we can write

$$(4.5) \quad W_e = \int_{\Omega_e} e_u \, dv = \iiint_{-\infty}^{\infty} e_u \, dv - \iiint_{V_p} e_u \, dv$$

Integration of the purely elastic strain energy density over the entire volume was done by Sneddon [16] and we will use it with our modification namely, multiplying the Sneddon's result by α^2 . We have

$$(4.6) \quad W_o = \iiint_{-\infty}^{\infty} e_u \, dv = \alpha^2 \iiint_{-\infty}^{\infty} o_u \, dv = \frac{8\alpha^2(1-\nu^2)p_o^2 l^3}{3E}.$$

Inserting expression (4.6) in equation (4.5) gives

$$(4.7) \quad W_e = W_o - \iiint_{V_p} e_u \, dv = W_o - \iiint_{V_p} I \, dv - \iiint_{V_p} II \, dv$$

where

$$(4.7a) \quad I = \frac{2}{3}(1+\nu) e \sigma_i^2 / 2E$$

$$II = \frac{3(1-2\nu)}{2E} e \sigma_m^2.$$

This is the strain energy contained within the elastic region.

Expressing the plastic strain energy in terms of its components we get

$$(4.8) \quad W_p = \iiint_{V_p} p_u \, dv = \iiint_{V_p} III \, dv + \iiint_{V_p} II \, dv$$

where

$$III = \frac{2}{3}(1+\nu) \left[Y^p \epsilon_1 - \frac{Y^2}{2E} \right]$$

(4.8a)

$$II = \frac{3(1-2\nu)}{2E} p \sigma_m^2 .$$

Now the total strain energy of the material is the sum of W_e as given by the equation (4.7) and W_p given by the equation (4.8). Adding equations (4.7) and (4.8) yields

$$(4.9a) \quad U = W_e + W_p = W_o - \iiint_{V_p} I \, dv - \iiint_{V_p} II \, dv + \iiint_{V_p} III \, dv + \iiint_{V_p} II \, dv$$

or shortly

$$(4.9b) \quad U = W_o - \iiint_{V_p} I \, dv + \iiint_{V_p} III \, dv .$$

The integrand I is given by the expression (4.7a) and upon inserting the proper function for the stress intensity $^e\sigma_1$ we get

$$(4.10) \quad I = \frac{1+\nu}{6E} \left(\frac{Y\alpha\lambda}{\pi} \right)^2 \frac{1}{\delta} \left[3\sin^2 \Psi + 4(1-2\nu)^2 \cos^2 \frac{\Psi}{2} \right].$$

The integrand III is given by the expression (4.8a) where $^p\epsilon_1$ denotes the plastic strain intensity which is defined as

$$(4.11) \quad ^p\epsilon_1 = \frac{1}{\sqrt{2(1+\nu)}} \left[(^p\epsilon_r - ^p\epsilon_\theta)^2 + (^p\epsilon_\theta - ^p\epsilon_z)^2 + (^p\epsilon_z - ^p\epsilon_r)^2 + \frac{3}{2} (^p\gamma_{r\theta}^2 + ^p\gamma_{rz}^2 + ^p\gamma_{\theta z}^2) \right]^{\frac{1}{2}}.$$

Inserting proper values of plastic strains in equation (4.11) and simplifying we get

$$(4.12) \quad ^p\epsilon_1 = \frac{Y\alpha\lambda}{2\pi E} (\delta)^{-\frac{1}{2}} \left[3\sin^2 \Psi + 4(1-2\nu)^2 \cos^2 \frac{\Psi}{2} \right]^{\frac{1}{2}}.$$

Now with the aid of the equation (4.12) we obtain integrand III in terms of the angle Ψ and the radius δ .

$$(4.13) \quad III = \frac{(1+\nu)Y^2}{3E} \left\{ \frac{\sqrt{2}\alpha\lambda}{\pi} (\delta)^{-\frac{1}{2}} \left[3\sin^2 \Psi + 4(1-2\nu)^2 \cos^2 \frac{\Psi}{2} \right] - 1 \right\}.$$

The boundaries of integration over the plastic region

are

$$\begin{aligned}
 (4.14) \quad & 0 \leq \delta \leq \delta_*(\psi) \\
 & 0 \leq \psi \leq 2\pi \\
 & 0 \leq \theta \leq 2\pi
 \end{aligned}$$

and the integral

$$\iiint_V f(r, z) \, dv = \iint f(r, z) \, 2\pi r \, dr \, dz$$

reduces to

$$2\pi \iint f(\delta, \psi) (1 + \delta \cos \psi) \, \delta \, d\delta \, d\psi .$$

This is done by substituting $1 + \delta \cos \psi$ for r (see figure (4.3)) and $2\pi(1 + \delta \cos \psi) \, \delta \, d\delta \, d\psi$ for dv .

$\theta = \text{constant plane}$

Ω_p is bounded by

$$\begin{aligned}
 0 & \leq \psi \leq 2\pi \\
 0 & \leq \delta \leq \delta_*(\psi)
 \end{aligned}$$

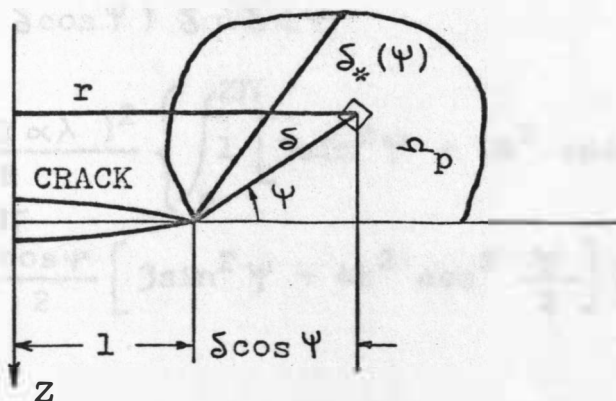


Figure 4.3

Let us define

$$1 - 2\gamma = k$$

$$\iiint_{\Omega_p} I \, dv = A = 2\pi \int_0^{2\pi} d\psi \int_0^{\delta_*(\psi)} I(\delta, \psi) [1 + \delta \cos \psi] \delta \, d\delta$$

$$(4.15) \quad \iiint_{\Omega_p} III \, dv = B = 2\pi \int_0^{2\pi} d\psi \int_0^{\delta_*(\psi)} III(\delta, \psi) [1 + \delta \cos \psi] \delta \, d\delta$$

$$\iiint_{\Omega_p} II \, dv = C = 2\pi \int_0^{2\pi} d\psi \int_0^{\delta_*(\psi)} II(\delta, \psi) [1 + \delta \cos \psi] \delta \, d\delta.$$

Therefore

$$A = 2\pi \int_0^{2\pi} \int_0^{\delta_*} \frac{1+\gamma}{6E} \left(\frac{\gamma \alpha \lambda}{\pi} \right)^2 \frac{1}{\delta} \left[3\sin^2 \psi + 4k^2 \cos^2 \frac{\psi}{2} \right]$$

$$(1 + \delta \cos \psi) \delta \, d\delta \, d\psi$$

$$(4.16) \quad A = \frac{(1+\gamma)(\gamma \alpha \lambda)^2}{3\pi E} \left\{ \int_0^{2\pi} 1 \left[3\sin^2 \psi + 4k^2 \cos^2 \frac{\psi}{2} \right] \delta_* \, d\psi + \int_0^{2\pi} \frac{\cos \psi}{2} \left[3\sin^2 \psi + 4k^2 \cos^2 \frac{\psi}{2} \right] \delta_*^2 \, d\psi \right\}.$$

Recalling the expression for δ_* from equation (3.26)

$$\delta_* = \left(\frac{\alpha\lambda}{\pi}\right)^2 \frac{1}{2} \left[3\sin^2\psi + 4k^2 \cos^2 \frac{\psi}{2} \right].$$

We substitute the equation (3.26) for δ_* in the equation (4.16) and continue the integration. Neglecting the terms which contain the parameter λ in powers of six or greater, the final result for A becomes

$$(4.17) \quad A = \frac{9(1+\nu)Y^2 l^3}{8\pi^2 E} (\alpha\lambda)^4 K(k) + O(\lambda^6)$$

where

$$(4.18) \quad K(k) = 1 + \frac{16}{9} k^2 + \frac{16}{9} k^4.$$

Similarly

$$B = 2\pi \int_0^{2\pi} \int_0^{\delta_*} \frac{(1+\nu)Y^2}{3E} \left\{ \frac{\sqrt{2\alpha\lambda}}{\pi} (\delta)^{-\frac{1}{2}} \left[3\sin^2\psi + 4k^2 \cos^2 \frac{\psi}{2} \right]^{\frac{1}{2}} - 1 \right\} (1 + \delta \cos \psi) \delta d\delta d\psi$$

$$\begin{aligned}
 (4.19) \quad B = \frac{2\pi(1+\nu)Y^2}{3E} & \left\{ \int_0^{2\pi} \left[\frac{2\sqrt{2}\alpha\lambda l^{3/2}}{3\pi} \delta_*^{3/2} (3\sin^2\psi + \right. \right. \\
 & \left. \left. 4k^2 \cos^2 \frac{\psi}{2} \right)^{\frac{1}{2}} - \frac{1}{2} \delta_*^2 \right] d\psi + \int_0^{2\pi} \left[\frac{2\sqrt{2}\alpha\lambda}{5\pi} \right. \\
 & \left. \left. (3\sin^2\psi + 4k^2 \cos^2 \frac{\psi}{2})^{\frac{1}{2}} \cos\psi \delta_*^{5/2} - \frac{\delta_*^3}{3} \cos\psi \right] d\psi \right\}.
 \end{aligned}$$

Substituting for δ_* from equation (3.26) in equation (4.19) and neglecting the terms which contain the parameter λ in the power of six or greater and upon integration the final result for B will be

$$(4.20) \quad B = \frac{15(1+\nu)Y^2 l^3}{16\pi^2 E} (\alpha\lambda)^4 K(k) + O(\lambda^6)$$

where $K(k)$ is given by the relation (4.18).

Now the total strain energy of the elastic-plastic solid surrounding a penny-shaped crack of radius l is obtained by inserting the equations (4.17) and (4.20) in the equation (4.9b). We have

$$U = W_0 - \frac{9(1+\nu')Y_1^2 l^3}{8\pi^2 E} (\alpha\lambda)^4 K(k) + \frac{15(1+\nu')Y_1^2 l^3}{16\pi^2 E} (\alpha\lambda)^4 K(k)$$

$$(4.21) \quad U = W_0 \left[1 - \frac{9\lambda^2 K(k)}{128\pi^2 (1-\nu')} \alpha^2 \right]$$

where

$$W_0 = \frac{8\alpha^2 (1-\nu'^2) p_0 l^3}{3E}.$$

The terms of orders higher than λ^4 have been neglected.

It is interesting to compare the amount of energy present in the elastic and plastic regions, i.e. W_e and W_p respectively. Recalling equations (4.7) and (4.8) we have

$$(4.22) \quad W_e = W_0 - A - C$$

$$W_p = B + C$$

or, upon substitution for A, B, and C

$$(4.23) \quad W_e = W_0 \left[1 - \frac{(27K(k) + 48(1+\nu')k(1+2k^2))}{64\pi^2 (1-\nu')} (\alpha\lambda)^2 \right]$$

$$W_p = W_0 \left[\frac{3(\alpha\lambda)^2}{8\pi^2 (1-\nu')} \right] \left[\frac{15K(k)}{16} + 2(1+\nu')k(1+2k^2) \right].$$

The ratio of energy contained in the plastic region to that present in the elastic region is

$$(4.24) \quad \frac{W_p}{W_e} \approx \frac{3(\alpha\lambda)^2 [15K(k) + 32(1+\nu)k(1+2k^2)]}{128\pi^2(1-\nu)}$$

or

$$(4.25) \quad \frac{W_p}{W_T} \approx \frac{3(\alpha\lambda)^2 [15K(k) + 32(1+\nu)k(1+2k^2)]}{128\pi^2(1-\nu)}.$$

Hence, it is seen that the energy contained within the plastic region is only a small part of the total strain energy (at least as long as the dimensionless load λ remains small).

It is noteworthy that the amount of energy dissipated during the yielding process prior to fracture is actually a little less than W_p (since only distortional energy can be dissipated). Therefore, we assumed that to find the dissipation D we have to subtract the volume energy contained in this region and the part of the distortional energy which is elastic from the energy in the plastic region W_p . Thus

$$(4.26) \quad D = W_p - \iiint_{\Omega_p} \frac{3(1-2\nu)}{2E} e \sigma_m^2 dv - \iiint_{\Omega_p} \frac{2(1+\nu)}{3} \frac{Y^2}{2E} dv$$

which upon carrying out the integration gives

$$(4.27) \quad D = \frac{3(1+\nu)Y^2 l^3 K(k)}{8\pi^2 E} (\alpha\lambda)^4 = \frac{9K(k)(\alpha\lambda)^2}{64\pi^2(1-\nu)} W_0.$$

By the same token the total amount of elastic strain energy in the solid surrounding a penny-shaped crack will be a little more than W_e , the excess being

$$(4.28) \quad \iiint_{\Omega_p} \frac{3(1-2\nu)}{2E} e \sigma_m^2 dv + \iiint_{\Omega_p} \frac{2(1+\nu)Y^2}{3E} dv.$$

Thus we have the total elastic strain energy

$$(4.29) \quad W_e^T = W_0 \left[1 - \frac{27 \lambda^2 K(k) \alpha^2}{128 \pi^2 (1-\nu)} \right].$$

It can be easily checked that the sum of energy dissipation D and the elastic energy W_e^T equals the total strain energy, as evaluated before. In fact we have

$$(4.30) \quad D + W_e^T = W_0 \left[1 - \frac{9 \lambda^2 K(k) \alpha^2}{128 \pi^2 (1-\nu)} \right]$$

which is identical to the equation (4.21) as expected.

Let us also notice that the ratios of the energy dissipation to the elastic energy

$$(4.30a) \quad \frac{D}{W_e^T} = \frac{2 \phi \lambda^2}{1-3 \phi \lambda^2} = 2 \phi \alpha^2 \lambda^2 \left[1 + 3 \phi \alpha^2 \lambda^2 \right]$$

and that of energy dissipation to the total strain energy

$$(4.30b) \quad \frac{D}{W_T} = \frac{2 \phi \lambda^2}{1 - \phi \lambda^2} = 2 \phi (\alpha \lambda)^2 \left[1 + \phi (\alpha \lambda)^2 \right]$$

have approximately the same numerical values as long as the dimensionless load λ remains small. In the above equations we have used a symbol ϕ to denote

$$\frac{9 \left[1 + \frac{16}{9}(1-2\nu)^2 + \frac{16}{9}(1-2\nu)^4 \right]}{128\pi^2(1-\nu)}$$

B. APPLICATION OF THE MINIMUM POTENTIAL ENERGY PRINCIPLE

The well-known principle of minimum potential energy may be stated as follows "the stable equilibrium state of a system is that for which the potential energy of the system attains the minimum". The potential energy of the system is equal to the strain energy of the system minus the change of potential due to the work done by external forces. If a new surface is created, a term allowing for the surface potential energy has to be added to the total potential energy. We shall discuss this case separately when we formulate the energy criterion for fracture. The strain energy of the system was derived in the last section and is given by equation (4.21) or (4.30).

The change of the potential energy due to the work of external forces on the system is given as

$$(4.31) \quad W = \int_S T_1 u_1^* ds$$

where T_1 is the force applied and u_1 is the kinematically admissible displacement field, and S denotes the part of the body surface on which the forces T_1 are applied.

The equation (4.31) can be written in the extended form as follows

$$(4.32) \quad W = \int_S (T_r e_{u_r} + T_\theta e_{u_\theta} + T_z e_{u_z}) ds$$

where S denotes the surface of the crack ($z=0$). Along this surface all the components of applied force are zero except the component in z direction which is equal to the applied pressure p_0 . The displacement u_z at $z=0$ can be found easily by multiplying Sneddon's result [16] by our unknown parameter α in agreement with our basic assumption. Since the Cauchy strain-displacement relations are linear, we can extend the assumption about the multiplication of the elastic strain by α into displacement field as well. Therefore

$$e_{u_i} = p_{u_i} = \alpha o_{u_i}$$

and

$$(4.33) \quad u_z(r) = e_{u_z}(r) = \alpha o_{u_z} = \alpha \frac{4(1-\nu^2)lp_0}{E} \sqrt{1 - \frac{r^2}{l^2}}.$$

Rewriting the equation (4.32) and inserting equation (4.33) for u_z we get

$$W = \int_S T_z u_z ds = 2 \int_0^1 p_0 u_z(r) 2\pi r dr$$

$$W = \frac{16(1-\nu^2)\alpha p_0^2 l}{E} \int_0^1 \sqrt{1 - \frac{r^2}{l^2}} \, r dr$$

$$(4.34) \quad W = \frac{16(1-\nu^2)\alpha p_0^2 l^3}{3E} .$$

Using equations (4.21) and (4.34), the potential energy of the system (I^*) becomes

$$(4.35) \quad I^* = U - W = W_0 \left[1 - \frac{9\lambda^2 K(k)}{128(1-\nu)\pi^2} \alpha^2 \right] - \frac{16(1-\nu^2)\alpha p_0^2 l^3}{3E} .$$

Recalling that

$$W_0 = \frac{8\alpha^2(1-\nu^2)p_0^2 l^3}{3E} ,$$

we can express the potential energy of the system in terms of W_0

$$(4.36) \quad I^* = W_0 \left[1 - \frac{2}{\alpha} - \frac{9\lambda^2 K(k)}{128\pi^2(1-\nu)} \alpha^2 \right] .$$

In order to minimize the potential energy of the system I^* we differentiate I^* with respect to the unknown parameter α and equate the result to zero. In order to differentiate,

we separate the unknown parameter α in the equation (4.36) which gives

$$(4.37) \quad I^* = \frac{8(1-\nu^2)p_0^2 l^3}{3E} \left[\alpha^2 - 2\alpha - \frac{9\lambda^2 K(k)}{128\pi^2(1-\nu)} \alpha^4 \right]$$

$$\frac{\partial I^*}{\partial \alpha} = \frac{8(1-\nu^2)p_0^2 l^3}{3E} \left[2\alpha - 2 - \frac{9\lambda^2 K(k)}{32\pi^2(1-\nu)} \alpha^3 \right] = 0$$

or

$$(4.38) \quad (\alpha - 1) - \frac{9\lambda^2 K(k)}{64\pi^2(1-\nu)} \alpha^3 = 0.$$

We will solve the cubic equation (4.38) for α by an approximation method. We expect α to be slightly greater than unity; then let us assume that it can be expressed in the form

$$(4.39) \quad \alpha = 1 + \alpha_1 \lambda^2 + \alpha_2 \lambda^4 + \dots$$

where the coefficients α_1 , and α_2 are to be subjected from the equation (4.38). Now, if we substitute this expression for α in the equation (4.38) and require that the coefficients of λ^2 and λ^4 be zero, we can solve for α_1 and α_2 . First we acquire expressions for α^2 and α^3 and neglect the terms which include λ in the power higher than four

$$\alpha^2 = 1 + 2\alpha_1 \lambda^2 + (\alpha_1^2 + 2\alpha_2) \lambda^4 + \dots$$

$$(4.40) \quad \alpha^3 = 1 + 3\alpha_1 \lambda^2 + 3(\alpha_1^2 + \alpha_2) \lambda^4 + \dots$$

Upon substitution of equations (4.39) and (4.40) in equation (4.38) and neglecting terms including λ to the power higher than four, we get

$$\alpha_1 \lambda^2 + \alpha_2 \lambda^4 - \frac{9K(k)}{64\pi^2(1-\nu)} (\lambda^2 + 3\alpha_1 \lambda^4) = 0$$

or

$$(4.41) \quad \left[\alpha_1 - \frac{9K(k)}{64\pi^2(1-\nu)} \right] \lambda^2 + \left[\alpha_2 - \frac{27K(k)}{64\pi^2(1-\nu)} \alpha_1 \right] \lambda^4 = 0.$$

Hence

$$(4.42) \quad \alpha_1 = \frac{9}{64} \frac{K(k)}{(1-\nu)\pi^2}$$

$$(4.43) \quad \alpha_2 = 3\alpha_1^2 = 3 \left[\frac{9K(k)}{64(1-\nu)\pi^2} \right]^2.$$

Finally, combining the last two equations with the equation (4.39) the expression for the parameter α reads as

$$(4.44) \quad \alpha = 1 + \frac{9K(k)}{64(1-\nu)\pi^2} \lambda^2 + 3 \left[\frac{9K(k)}{64(1-\nu)\pi^2} \right]^2 \lambda^4 + \dots$$

C. RELATION BETWEEN THE APPLIED LOAD AND THE DISPLACEMENT
AT THE CENTER OF THE CRACK

Equation (4.33) of the last section describes the displacement at the surface of the crack ($z = 0$). In this section we will illustrate the relationship between the applied load p_0 and the displacement at the center of the crack where $r = 0$ (see figure 4.4).

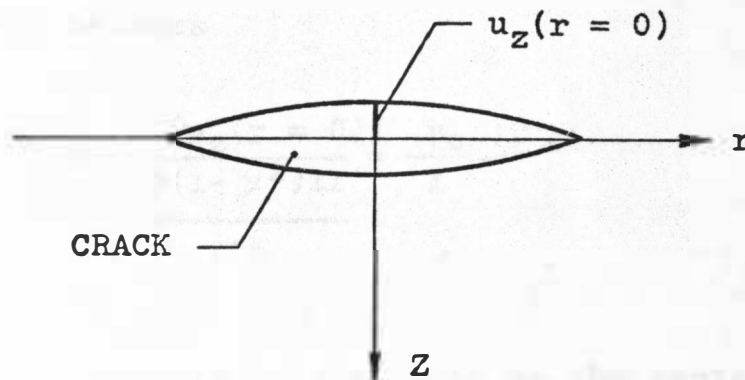


Figure 4.4

Displacement at the center of the crack for a purely elastic solid is given by Sneddon [16] as

$$(4.45) \quad \left. \frac{\partial u_z}{\partial r} \right|_{r=0} = \frac{4(1-\nu^2)lp_0}{\pi E}$$

and for an elastic-plastic solid is

$$(4.46) \quad u_z \Big|_{r=0} = \frac{4(1-\nu^2)l\alpha}{\pi E} p_0 .$$

From equation (4.45) it is seen that the displacement at the center of the crack for a purely elastic solid is linearly related to the applied load p_0 . Now in order to graph this relation, we actually plot a dimensionless displacement versus a dimensionless load $\lambda = p_0 / Y$. Hence equation (4.45) becomes

$$(4.47) \quad \omega = \frac{u_z(r=0)}{\pi E} = \frac{p_0}{Y}$$

The elastic-plastic displacement at the center of the crack depends on the parameter α and the parameter α in turn depends on the applied load p_0 ; so the relation between the displacement and the applied load is not linear as it was for the purely elastic case. The expression for parameter α was also derived in the last section and is given as

$$(4.48) \quad \alpha = 1 + 2\phi \left(\frac{p_0}{Y}\right)^2 + 12\phi^2 \left(\frac{p_0}{Y}\right)^4 + \dots$$

Since p_0 / Y is small compared to unity, we will neglect the

term $(p_0 / Y)^4$ and substitute equation (4.48) in equation (4.46). After some mathematical manipulation the relation between the dimensionless load and the dimensionless displacement becomes

$$(4.49) \quad e_w = p_w = \frac{u_z(r=0)}{\frac{4(1-\nu^2)lY}{\pi E}} = \left(\frac{p_0}{Y}\right) + 2\phi \left(\frac{p_0}{Y}\right)^3$$

Recalling the expression for

$$\phi = \frac{9 \left[1 + \frac{16}{9} (1-2\nu)^2 + \frac{16}{9} (1-2\nu)^4 \right]}{128 \pi^2 (1-\nu)}$$

we calculate ϕ for $\nu = 0$. The result of this calculation gives $\phi = 0.032$. Now equation (4.49) can be written as

$$(4.50) \quad e_w = p_w = \left(\frac{p_0}{Y}\right) + 0.064 \left(\frac{p_0}{Y}\right)^3 .$$

For plotting, we have chosen values of p_0 / Y between 0.5 and 1.0; figure 8 illustrates the relation between the applied load and the displacement at the center of the crack for purely elastic and elastic-plastic solids. Although the differences are minor, this plot indicates one of the general features of an elastic-plastic solution.

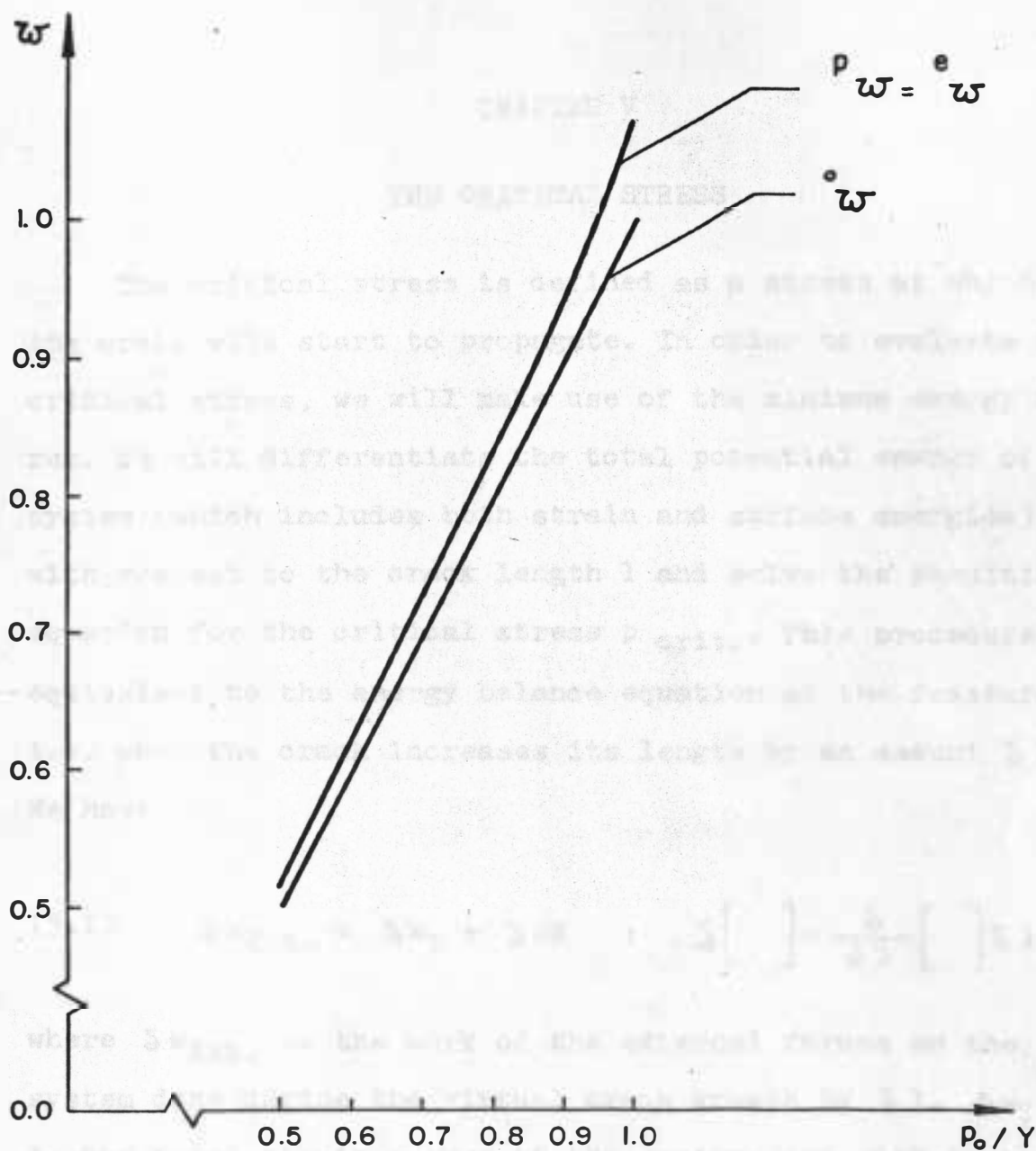


FIG. 8 ELASTIC AND ELASTIC-PLASTIC DISPLACEMENT AT THE CENTER OF THE CRACK AS A FUNCTION OF THE APPLIED LOAD.

CHAPTER V

THE CRITICAL STRESS

The critical stress is defined as a stress at which the crack will start to propagate. In order to evaluate the critical stress, we will make use of the minimum energy theorem. We will differentiate the total potential energy of the system (which includes both strain and surface energies) with respect to the crack length l and solve the resulting equation for the critical stress $p_{crit.}$. This procedure is equivalent to the energy balance equation at the fracture, i.e. when the crack increases its length by an amount δl . We have

$$(5.1) \quad \delta W_{Ext.} = \delta W_T + \delta SE \quad ; \quad \delta \left[\quad \right] = \frac{\delta}{\delta l} \left[\quad \right] \delta l$$

where $\delta W_{Ext.}$ is the work of the external forces on the system done during the virtual crack growth by δl , δW_T is the total strain energy of the system, and δSE is the surface energy both associated with the virtual crack growth.

The first term on the right hand side of the equation (5.1), i.e. the increment of the total strain energy of the system can be obtained from the equation (4.21) derived in

the last chapter. Let us recall

$$(5.2) \quad W_T = U = W_0 \left[1 - \frac{9 \lambda^2 K(k)}{128 \pi^2 (1-\nu)} \alpha^2 \right].$$

The work of the external forces done during the virtual crack growth is evaluated as the work of the applied pressure on the displacement difference (see figure 5.1)

$$(5.3) \quad e_{u_z}(r, l + \delta l) - e_{u_z}(r, l).$$

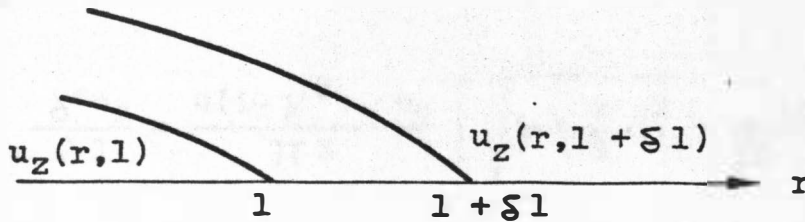


Figure 5.1

Thus multiplied by p_0 and integrated over the crack surface yields

$$(5.4) \quad \delta W_{\text{Ext.}} = \int_S p_0 \left[e_{u_z}(r, l + \delta l) - e_{u_z}(r, l) \right] 2\pi r dr$$

or, upon observing that

$$(5.5) \quad e_{u_z}(r, 1 + \delta l) - e_{u_z}(r, 1) = \frac{\partial e_{u_z}(r, 1)}{\partial l} \delta l$$

we have

$$(5.6) \quad W_{\text{Ext.}} = 2 \int_0^1 p_0 \frac{\partial e_{u_z}(r, 1)}{\partial l} \delta l 2\pi r dr.$$

Recalling that

$$e_{u_z} = \frac{4(1-\nu^2)}{\pi E} l \propto p_0 \sqrt{1 - \left(\frac{r}{l}\right)^2}$$

and with

$$(5.7) \quad \frac{\partial e_{u_z}}{\partial l} = \frac{4(1-\nu^2) \propto p_0}{\pi E} \left[\sqrt{1 - \left(\frac{r}{l}\right)^2} + \frac{(r/l)^2}{\sqrt{1 - (r/l)^2}} \right];$$

we obtain

$$(5.8) \quad \delta W_{\text{Ext.}} = \frac{16(1-\nu^2)p_0^2 \propto}{E} \delta l \int_0^1 \left[\sqrt{1 - \left(\frac{r}{l}\right)^2} + \frac{(r/l)^2}{\sqrt{1 - (r/l)^2}} \right] r dr$$

which reduces to

$$(5.9) \quad \delta W_{\text{Ext.}} = \frac{16(1-\nu^2)p_0^2 \propto}{E} \delta l.$$

The surface energy of the system due to the existence of crack may be expressed as a product of the specific energy of

the solid and the area of the crack. The surface area of the crack consists of two circles each with a radius equal to 1. Hence the surface energy is

$$(5.10) \quad SE = 2\pi l^2 \gamma$$

where γ denotes the specific surface energy.

Substituting equations (5.10), (5.9) and (5.2) in the relation (5.1) and writing the new relationship in terms of W_0 which is given by the equation (4.6) we get

$$(5.11) \quad \frac{16(1-\nu^2)p_0^2 l^2 \alpha}{E} = \frac{\partial}{\partial l} \left\{ W_0 \left[1 - \frac{9\lambda^2 K(k) \alpha^2}{128\pi^2(1-\nu)} \right] \right\} + \frac{\partial}{\partial l} (2\pi l^2 \gamma).$$

Substituting for W_0 from the equation (4.6), differentiating with respect to the crack length l , and solving for p_0 gives

$$(5.12) \quad p_{\text{crit.}} = \sqrt{\frac{E\gamma\pi}{2(1-\nu^2)l}} \left[2\alpha - \alpha^2 + \frac{9K(k)\lambda^2}{128(1-\nu)\pi^2} \alpha^4 \right]^{-\frac{1}{2}}.$$

This is an expression for the critical stress of a penny-shaped crack in an elastic-plastic solid. The expression is implicit since pressure p_0 is also included in α .

Let us denote the quantity $\sqrt{\frac{\pi E \gamma}{2(1-\nu^2)l}}$ by p_G and call it "Griffith critical stress". In fact it was derived by Sack [15] and Sneddon [16] for a purely elastic solid. With

this notation equation (5.12) becomes

$$(5.13) \quad p_{\text{crit.}} = p_G \left[2\alpha - \alpha^2 + \frac{9 K(k) \lambda^2}{128(1-\nu)\pi^2} \alpha^4 \right]^{-\frac{1}{2}},$$

where α is a function of the applied load λ ($\lambda = p_0 / Y$).

According to equation (4.44) we have

$$(5.14) \quad \alpha = 1 + 2\phi\lambda^2 + 12\phi^2\lambda^4 + \dots$$

where for simplicity we have named

$$(5.15) \quad \phi = \frac{9 K(k)}{128(1-\nu)\pi^2}$$

where

$$K(k) = 1 + \frac{16}{9}(1-2\nu)^2 + \frac{16}{9}(1-2\nu)^4.$$

To evaluate the expression in the bracket of equation (5.13)

we substitute respectively

$$2\alpha = 2 + 4\phi\lambda^2 + 24\phi^2\lambda^4 + \dots$$

$$(5.16) \quad -\alpha^2 = -1 - 4\phi\lambda^2 - 28\phi^2\lambda^4 + \dots$$

$$\alpha^4 = 1 + 8\phi\lambda^2 + 72\phi^2\lambda^4 + \dots$$

Finally, neglecting terms which include λ to the powers higher than four, the expression in the bracket of equation (5.13) is written as

$$\left[1 + \phi \lambda^2 + 4 \phi^2 \lambda^4 + \dots \right]^{-\frac{1}{2}} .$$

If we divide both sides of the equation (5.13) by the yield stress Y and call $p_G / Y = \lambda_G$ and $p_0 / Y = \lambda$, we obtain

$$(5.17) \quad \frac{\lambda}{\lambda_G} = \left[1 + \phi \lambda^2 + 4 \phi^2 \lambda^4 + \dots \right]^{-\frac{1}{2}} .$$

Expanding the right side of equation (5.17) into Maclaurin series and designating λ / λ_G by X , equation (5.17) becomes

$$(5.18) \quad X = 1 - \frac{\phi \lambda_G^2}{2} X^2$$

where the ratio of the Griffith stress to the yield point stress is considered to be small ($\lambda_G = p_G / Y \ll 1$).

We expect X to be equal to $1 + \varepsilon$, where ε is a small correction quantity. Inserting $1 + \varepsilon$ for X and $1 + 2\varepsilon + \dots$ for X^2 in equation (5.18) and solving for ε we have

$$(5.19) \quad \varepsilon = \frac{-\phi \lambda_G^2}{2(1 + \phi \lambda_G^2)} \approx -\frac{\phi \lambda_G^2}{2} .$$

Substituting for ε from equation (5.19), the final result for $p_{crit.}$ is obtained as

$$(5.20) \quad p_{\text{crit.}} = p_G X = p_G (1 + \varepsilon)$$

or

$$(5.20a) \quad p_{\text{crit.}} = p_G \left[1 - \frac{\phi \lambda_G^2}{2} \right].$$

Equation (5.20a) is the final expression derived here for the critical stress in an elastic-plastic solid containing a penny-shaped crack. It is seen that our $p_{\text{crit.}}$ differs from that obtained by Sack and Sneddon by a certain correction factor

$$(5.20b) \quad \varepsilon = \frac{9+16(1-2\nu)^2+16(1-2\nu)^4}{128(1-\nu)\pi^2} \frac{\pi E \gamma}{4(1-\nu^2)lY^2}.$$

In figure 9 we graph the critical stress versus the crack length to illustrate the effect of the crack length upon the critical stress required to start the propagation of crack. However, before proceeding, let us perform some transformations on the equation (5.20b) in order to be able to graph the dimensionless critical load $p_{\text{crit.}} / Y$ versus dimensionless crack length l / l_* . First, let us write equation (5.20a) in the form

$$(5.21) \quad \frac{p_{\text{crit.}}}{Y} = \frac{p_G}{Y} \left[1 - \frac{\phi \lambda_G^2}{2} \right]$$

and observe that

$$(5.22) \quad \frac{\phi}{2} \left(\frac{p_G}{Y} \right)^2 = \frac{\phi}{2Y^2} p_G^2 = \frac{\phi}{2\pi Y^2} \frac{K_C^2}{1}$$

where K_C denotes the Irwin [10] stress concentration factor.

For our geometry

$$K_C = p_G \sqrt{\pi l} .$$

It can be easily seen that a quantity $K_C^2 / 2\pi Y^2$ has a dimension of length. Let us choose

$$l_* = \frac{K_C^2}{2\pi Y^2}$$

as a normality entity for the crack length. With this, the equation (5.22) simplifies to

$$(5.23) \quad \frac{\phi}{2} \left(\frac{p_G}{Y} \right)^2 = \phi \frac{l_*}{1} .$$

Similarly we can write p_G / Y as

$$\frac{p_G}{Y} = \frac{\sqrt{2} K_C}{\sqrt{2\pi l} Y}$$

which is simplified to

$$(5.24) \quad \frac{p_G}{Y} = \sqrt{2} \left(\frac{l_*}{1} \right)^{\frac{1}{2}} .$$

Let us now introduce a symbol ξ for the dimensionless crack length l / l_* . With the aid of equations (5.24) and (5.23), equation (5.21) becomes

$$(5.25) \quad \frac{p_{crit.}}{Y} = \sqrt{\frac{2}{\xi}} \left[1 - \frac{\phi}{\xi} \right]$$

This equation also differs from the result derived from the Griffith-Irwin theory of fracture for the case of a penny-shaped crack (as given by (5.24)) only by a correction factor ϕ/ξ which has a substantial contribution for $\xi \rightarrow 1$ only i.e., for the crack length approaching the "characteristic" value $l_* = K_c^2 / 2\pi Y^2$, which is in agreement with the result of Wnuk [12] derived from the Barenblatt-Dugdale model of the crack.

Figure 9 shows the graphs of p_G / Y and $p_{crit.} / Y$ versus the dimensionless crack length ξ . The quantity ϕ which depends only on the Poisson's ratio of the material is evaluated for $\nu = 0.3$.

It should be noted that for extremely small crack length ($\xi \rightarrow 0$), our correction factor which is proportional to ξ^{-1} becomes infinite. This range of ξ however, is not included in the present approach; as for very small crack length, the critical stress tends to yield point and thus the requirement $(p_{crit.} / Y) \ll 1$ is not satisfied. From figure 9, it is seen that for ξ below 0.039 equation (5.25) may not be used.

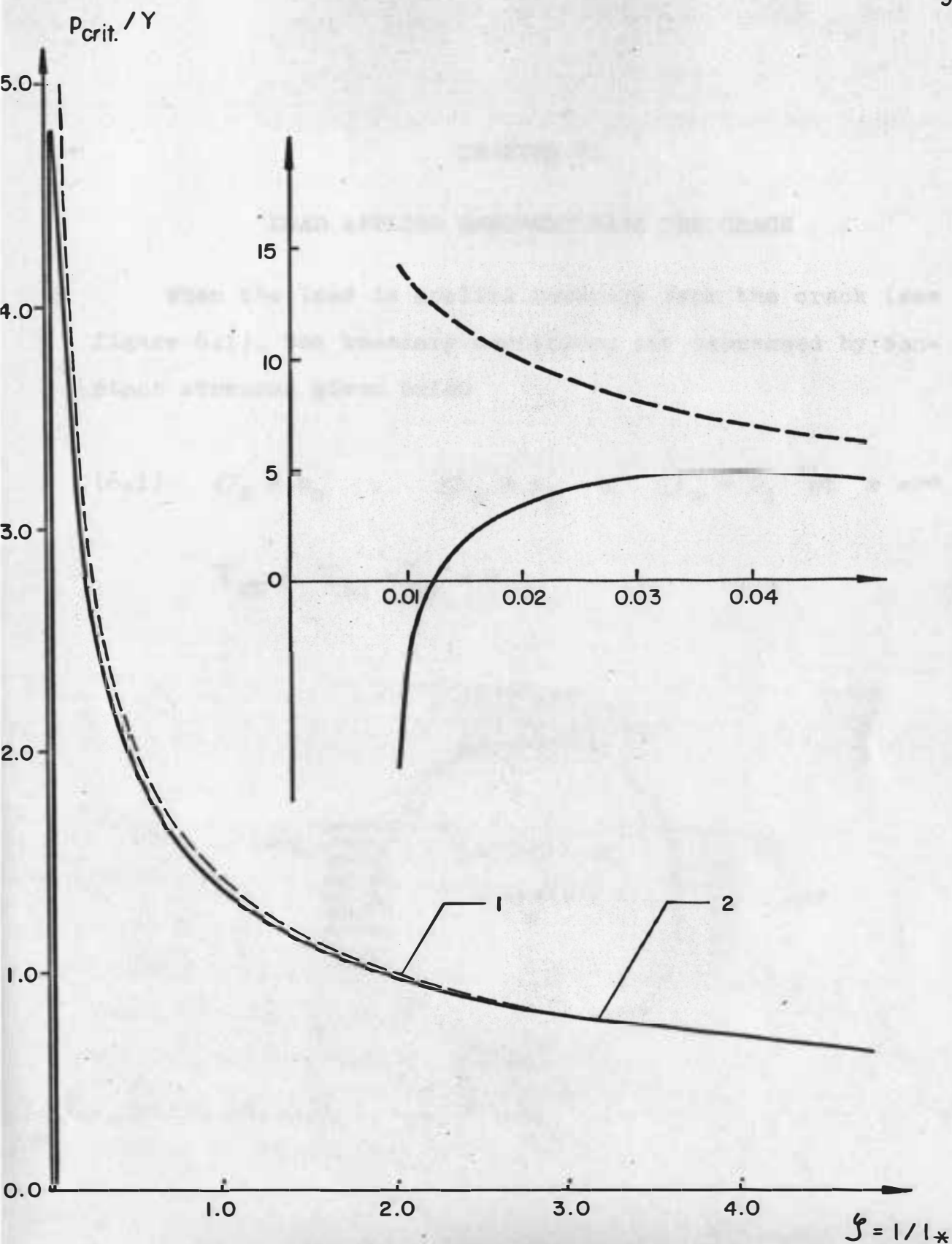


FIG.9 THE CRITICAL STRESS VS. THE DIMENSIONLESS CRACK LENGTH

1. Griffith - Irwin theory
2. Present paper (the descending part of the curve for $\mathcal{G} \rightarrow 0$ is beyond the limits of applicability). The region of small \mathcal{G} has been magnified.

CHAPTER VI

LOAD APPLIED REMOTELY FROM THE CRACK

When the load is applied remotely from the crack (see figure 6.1), the boundary conditions are expressed by constant stresses given below

$$(6.1) \quad \sigma_z = p_0, \quad \sigma_r = p_1, \quad \sigma_\theta = p_1 \quad \text{at} \quad r = \infty$$

$$\tau_{r\theta} = \tau_{rz} = \tau_{\theta z} = 0.$$

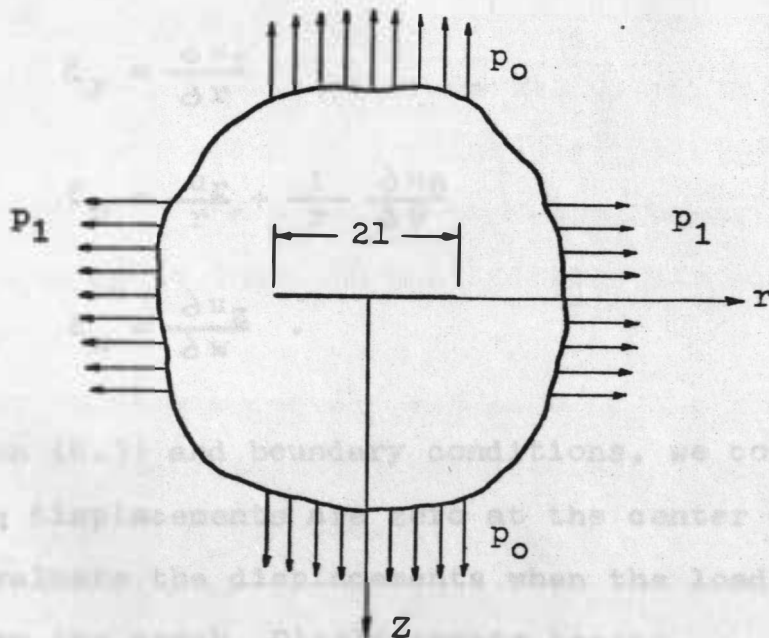


Figure 6.1

Using Hooke's law in cylindrical coordinate system, we obtain the strains for the figure 6.1. The strains become

$$\begin{aligned}
 \varepsilon_z &= \frac{1}{E} [p_0 - 2\nu p_1] \\
 \varepsilon_r &= \frac{1}{E} [p_1 - \nu (p_0 + p_1)] \\
 \varepsilon_\theta &= \frac{1}{E} [p_1 - \nu (p_0 + p_1)] \\
 \gamma_{r\theta} &= \gamma_{rz} = \gamma_{\theta z} = 0 .
 \end{aligned}
 \tag{6.2}$$

Strain-displacement relations in the cylindrical coordinate system are

$$\begin{aligned}
 \varepsilon_r &= \frac{\partial u_r}{\partial r} \\
 \varepsilon_\theta &= \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \\
 \varepsilon_z &= \frac{\partial u_z}{\partial z} .
 \end{aligned}
 \tag{6.3}$$

From equation (6.3) and boundary conditions, we conclude that all the displacements are zero at the center of the crack. We evaluate the displacements when the load is applied remotely from the crack. Displacements become

$$\begin{aligned}
 u_r &= \frac{1}{E} \left[p_1 - \nu (p_0 + p_1) \right] r \\
 (6.4) \quad u_z &= \frac{1}{E} \left[p_0 - 2\nu p_1 \right] z \\
 u_\theta &= 0 .
 \end{aligned}$$

Superposing the above distributions with the stresses, strains, and displacements correspondingly induced by the pressurized crack would lead us to a solution of the crack problem in which the stresses are applied at infinity while the surface of the crack is traction free (note that along the crack surface $\sigma_z = -p_0$ for pressurized crack cancels out with $\sigma_z = p_0$ prescribed by (6.1)). This can be done in the elastic range without much difficulty since, the superposition principle is valid here and the solution to the pressurized crack is given by Sneddon. In the elastic-plastic range, however, even the simplest procedure based upon the variational principle leads to considerable mathematical difficulties.

In order to give a certain estimate of the size of the plastic zone, we shall assume a pseudo-elastic approach based on the following hypotheses:

- 1) Within the elastic region the principle of superposition remains valid.
- 2) Strains and displacements for an elastic-plastic

problem differ from those resulting in the theory of elasticity only by a certain factor α . Our previous analysis shows this factor is only slightly larger than one for small ratios of load to yield stress, and therefore here we shall assume $\alpha = 1$.

- 3) The stresses in the elastic region obey the Hooke's law while in the plastic region a different physical relations have to be used, e.g. Hencky-Ilyushin theory of small plastic deformations.

Even under these restrictions the mathematical complexity of the problem faced here does not allow us to follow the lines of the first five chapters because, Sneddon's asymptotic formulae contain only one leading term of order $\xi^{-\frac{1}{2}}$ and this appears to be an unsatisfactory approximation for the superposed problem.

Therefore we shall confine our interest only to the crack plane ($z = 0$) and with the assumptions 1, 2, and 3 we shall proceed to determine the length of the plastic zone along the crack plane for two cases namely, that of pressurized crack and that of loadings applied remotely from the crack. We shall use here complete Sneddon's equations for stresses in front of the crack rather than applying the asymptotic expansions (as we did in the previous chapters).

For $z = 0$ and $\rho > 1$ according to Sneddon [16] the stresses are

$$\begin{aligned}
 {}^e\sigma_z &= \frac{2p_0}{\pi} \left[\frac{1}{\sqrt{\rho^2-1}} - \sin^{-1} \frac{1}{\rho} \right] \\
 {}^e\sigma_r &= \frac{2p_0}{\pi} \left[\frac{1}{\sqrt{\rho^2-1}} - \left(\nu + \frac{1}{2} \right) \sin^{-1} \frac{1}{\rho} \right] \\
 {}^e\sigma_\theta &= \frac{2p_0}{\pi} \left[\frac{2}{\sqrt{\rho^2-1}} - \left(\nu + \frac{1}{2} \right) \sin^{-1} \frac{1}{\rho} \right] \\
 {}^e\tau_{r\theta} &= {}^e\tau_{\theta z} = {}^e\tau_{rz} = 0
 \end{aligned}
 \tag{6.5}$$

where ρ is the ratio of coordinate r and the crack radius l .

To determine the scope of plastic region for the pressurized crack we will use the Huber-Mises-Hencky plasticity condition. First we evaluate the stress intensity

$$\begin{aligned}
 {}^e\sigma_1 &= \frac{1}{\sqrt{2}} \left[({}^e\sigma_r - {}^e\sigma_\theta)^2 + ({}^e\sigma_\theta - {}^e\sigma_z)^2 + ({}^e\sigma_z - {}^e\sigma_r)^2 \right. \\
 &\quad \left. + 6({}^e\tau_{rz}^2 + {}^e\tau_{r\theta}^2 + {}^e\tau_{\theta z}^2) \right]^{\frac{1}{2}}.
 \end{aligned}
 \tag{6.6}$$

Substituting proper values of stresses from equation (6.5) in equation (6.6) and equating the stress intensity to yield stress Y of the solid we have

$$\begin{aligned}
 Y^2 &= \left(\frac{2p_0}{\pi} \right)^2 (1-2\nu)^2 \left[\frac{1}{\rho_*^2-1} + \frac{1}{4} \left(\sin^{-1} \frac{1}{\rho_*} \right)^2 \right. \\
 &\quad \left. - \frac{\sin^{-1} \frac{1}{\rho_*}}{2\sqrt{\rho_*^2-1}} \right].
 \end{aligned}
 \tag{6.7}$$

Denoting p_0 by $Y\lambda_0$ where λ_0 is the dimensionless load and solving equation (6.7) for λ_0 in terms of ρ_* we get

$$(6.8) \quad \lambda_0 = \frac{\pi}{2(1-2\nu)} \left[\frac{1}{\rho_*^2 - 1} + \frac{1}{4} (\sin^{-1} \frac{1}{\rho_*})^2 - \frac{\sin^{-1} \frac{1}{\rho_*}}{2\sqrt{\rho_*^2 - 1}} \right]^{-\frac{1}{2}} = F(\rho_*).$$

In equation (6.8) ρ_* denotes the scope of plastic region for the angle Ψ being zero. Figure 10 illustrates the scope of plastic zone under various values of λ_0 for values of Poisson's ratio equal to 0, 0.3, and 0.5 respectively. Note that since we are using the complete expressions for stresses, we are not restricted to small values of λ_0 like we were in the earlier chapter. For comparison the dotted lines show the corresponding results which follow from our small-scale-yielding equation (see chapters 3 and 4).

Following a similar line of argument, the stresses for the superposed problem are

$$(6.9) \quad \begin{aligned} \sigma_z &= p_0 + {}^e\sigma_z = p_0 + \frac{2p_0}{\pi} \left[\frac{1}{\sqrt{\rho^2 - 1}} - \sin^{-1} \frac{1}{\rho} \right] \\ \sigma_r &= p_1 + {}^e\sigma_r = p_1 + \frac{2p_0}{\pi} \left[\frac{1}{\sqrt{\rho^2 - 1}} - \left(\nu + \frac{1}{2} \right) \sin^{-1} \frac{1}{\rho} \right] \\ \sigma_\theta &= p_1 + {}^e\sigma_\theta = p_1 + \frac{2p_0}{\pi} \left[\frac{2}{\sqrt{\rho^2 - 1}} - \left(\nu + \frac{1}{2} \right) \sin^{-1} \frac{1}{\rho} \right] \\ \tau_{r\theta} &= \tau_{\theta z} = \tau_{rz} = 0. \end{aligned}$$

Applying the Huber-Mises-Hencky plasticity condition we get

$$(6.10) \quad Y^2 = \left(\frac{2p_0}{\pi}\right)^2 (1-2\nu)^2 G(\rho_*) + \frac{p_0(p_1-p_0)}{\pi} (1-2\nu) H(\rho_*) + (p_1 - p_0)^2$$

where

$$G(\rho_*) = \frac{1}{\rho_*^2-1} + \frac{1}{4} \left(\sin^{-1} \frac{1}{\rho_*}\right)^2 - \frac{\sin^{-1} \frac{1}{\rho_*}}{2\sqrt{\rho_*^2-1}}$$

$$H(\rho_*) = \sin^{-1} \frac{1}{\rho_*} - \frac{1}{2\sqrt{\rho_*^2-1}}.$$

It is interesting to note that when $p_1 = p_0$ equation (6.10) reduces to equation (6.7). Which shows that when the solid is under the influence of hydrostatic stresses the scope of plastic region is not different from that obtained for the case where the load is applied at the crack surface. At least it follows from the Huber-Mises-Hencky plasticity condition which ignores the influence of the hydrostatic stresses.

Denoting p_1 by $Y\lambda_1$, where λ_1 is also a dimensionless load, we shall determine the scope of the plastic region under various conditions of loading. When $\lambda_0 = 1/2$ equation (6.10) can be solved for λ_1 in terms of ρ_* ; the result becomes

$$(6.11) \quad \lambda_1 = \frac{1}{2} - \frac{(1-2\nu)}{4\pi} H(\rho_*) \pm \sqrt{1 + \frac{(1-2\nu)^2}{\pi^2} \left[\frac{H^2(\rho_*)}{16} - G(\rho_*) \right]}$$

Figure 11-a illustrates the scope of the plastic region for various values of Poisson's ratio.

When $\lambda_1=0$ equation (6.10) can be solved for λ_0 in terms of ρ_* ; the result is

$$(6.12) \quad \lambda_0 = \left[\frac{4(1-2\nu)^2}{\pi^2} G(\rho_*) - \frac{(1-2\nu)}{\pi} H(\rho_*) + 1 \right]^{-\frac{1}{2}}.$$

Figure 11-b illustrates the scope of the plastic region for various values of Poisson's ratio.

It is interesting to note the affect of the compressive stresses on the size of plastic zone : it is seen from figure 11-a that the compressive loads ($\sigma_\theta = -p_1$ and $\sigma_r = -p_1$) applied remotely from the crack do indeed influence the yielding at the crack tip. Comparison of magnitudes reveal that it takes only about 1/3 of the load required to produce large plastic zones in tension to achieve the same result in compression.

Figure 12 illustrates the comparison between our solution and the solution of Olesiak and Wnuk [12] resulting from analysis based on the Dugdale model.

Figure 13 shows the growth of plastic zone with the increasing load for two cases of loading considered here namely, pressure applied at the crack surface and tension applied

remotely from the crack. Of course, here we assume that the scope of the plastic region along the crack surface is a representative measure of the size of the yielded zone.

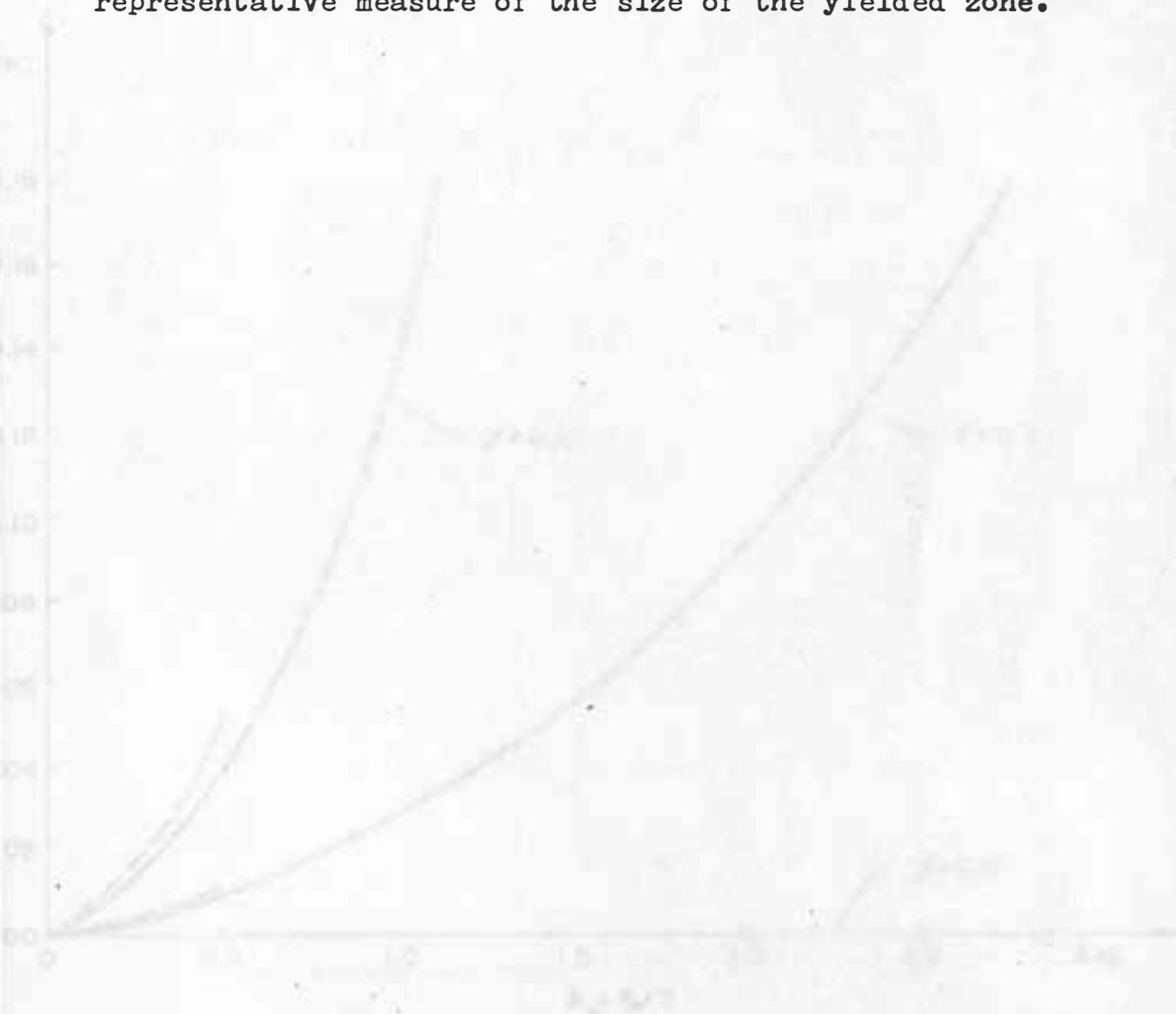
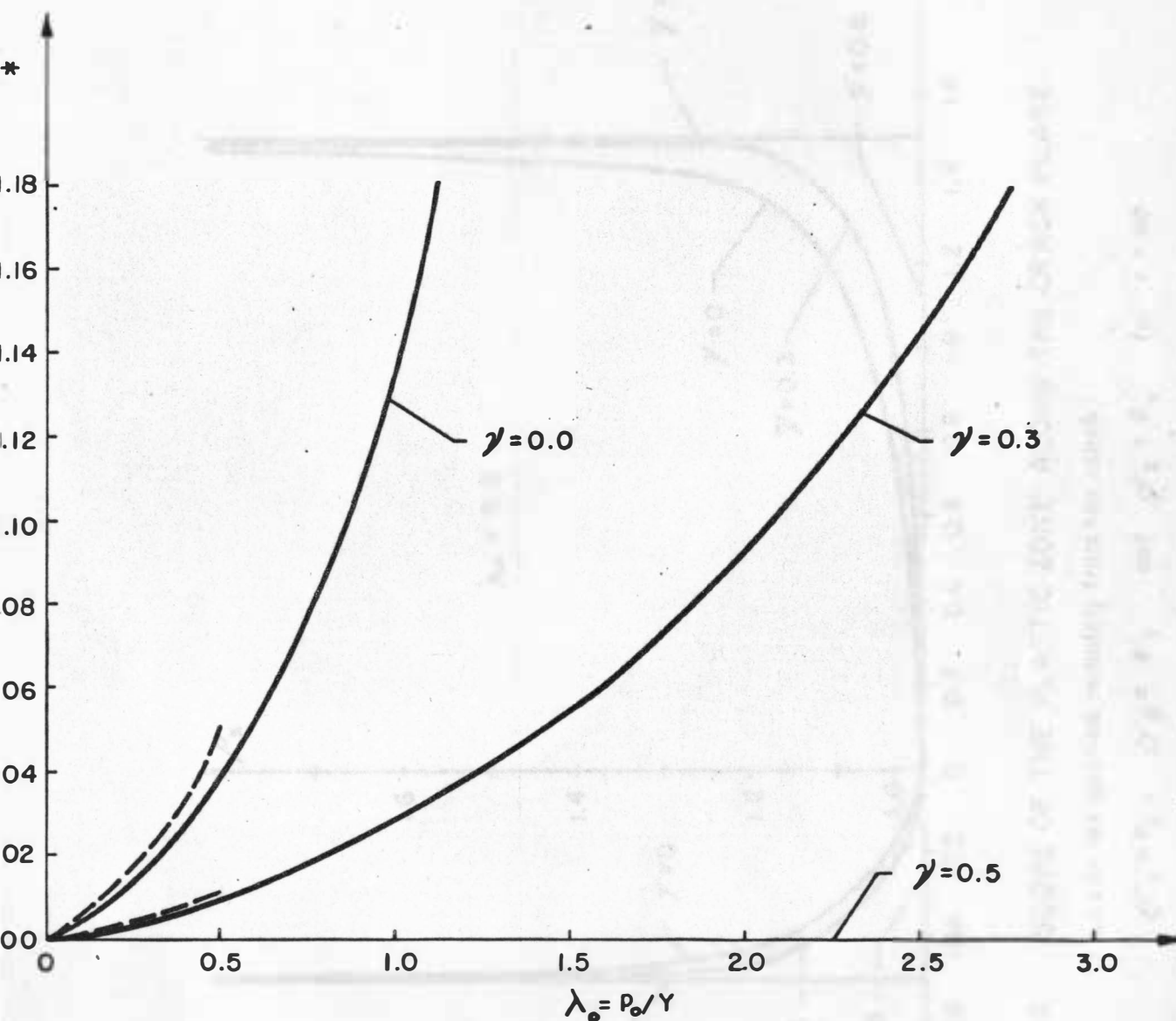


FIG 10 PROFILE OF THE PLASTIC REGION ALONG THE CRACK, $\nu = 0.1$

SOURCE: REPLY GIVES THE RESULT WHEN USING THE ENERGY BALANCE

PRESSURIZED CRACK

FIG.10 SCOPE OF THE PLASTIC REGION ALONG THE CRACK PLANE ($z=0$)

Dotted lines denote the result which follows from leading term solution.

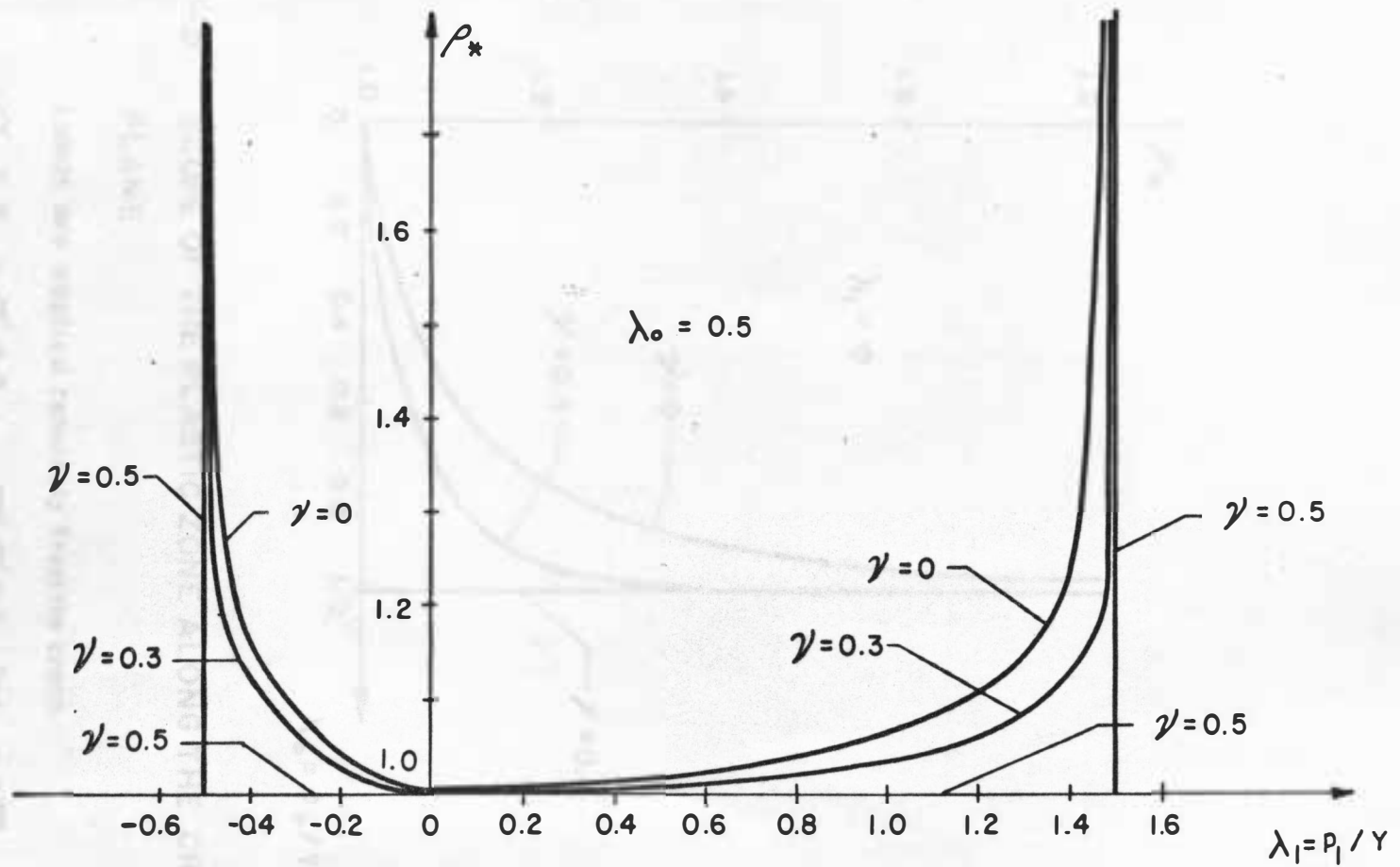


FIG. II - a SCOPE OF THE PLASTIC ZONE ALONG THE CRACK PLANE.

Loads are applied remotely from the crack :

$$\sigma_r = p_1, \quad \sigma_\theta = p_1 \quad \text{and} \quad \sigma_z = p_0 \quad \text{for} \quad r = \infty$$

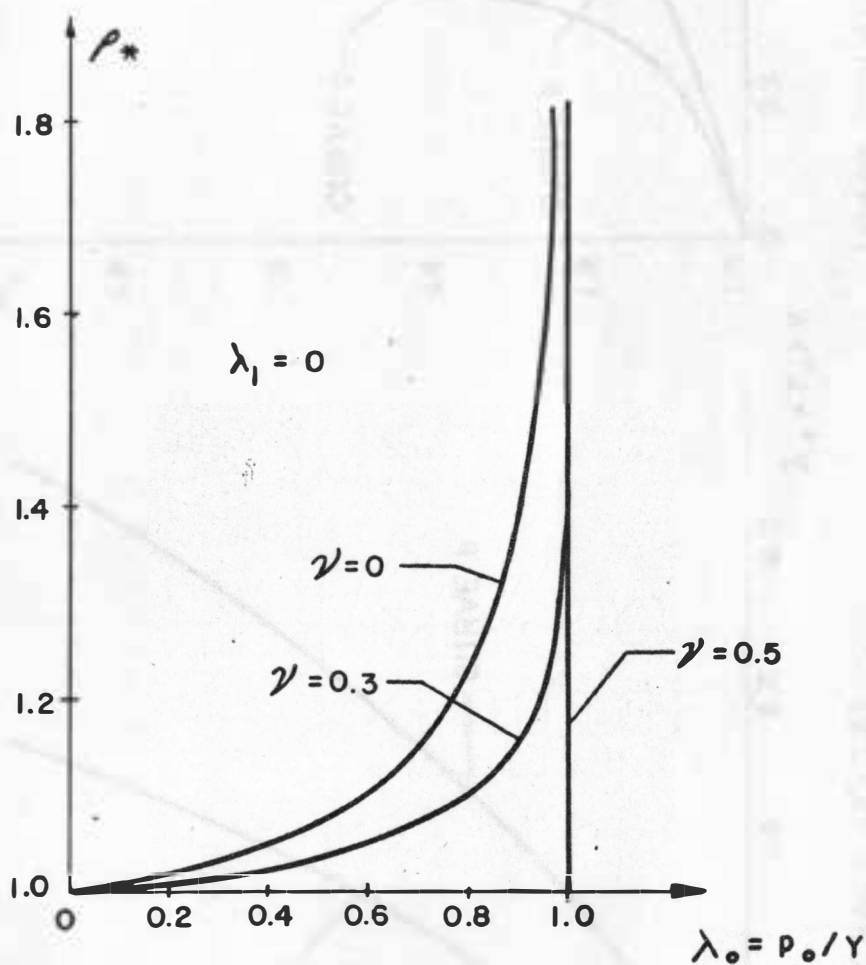
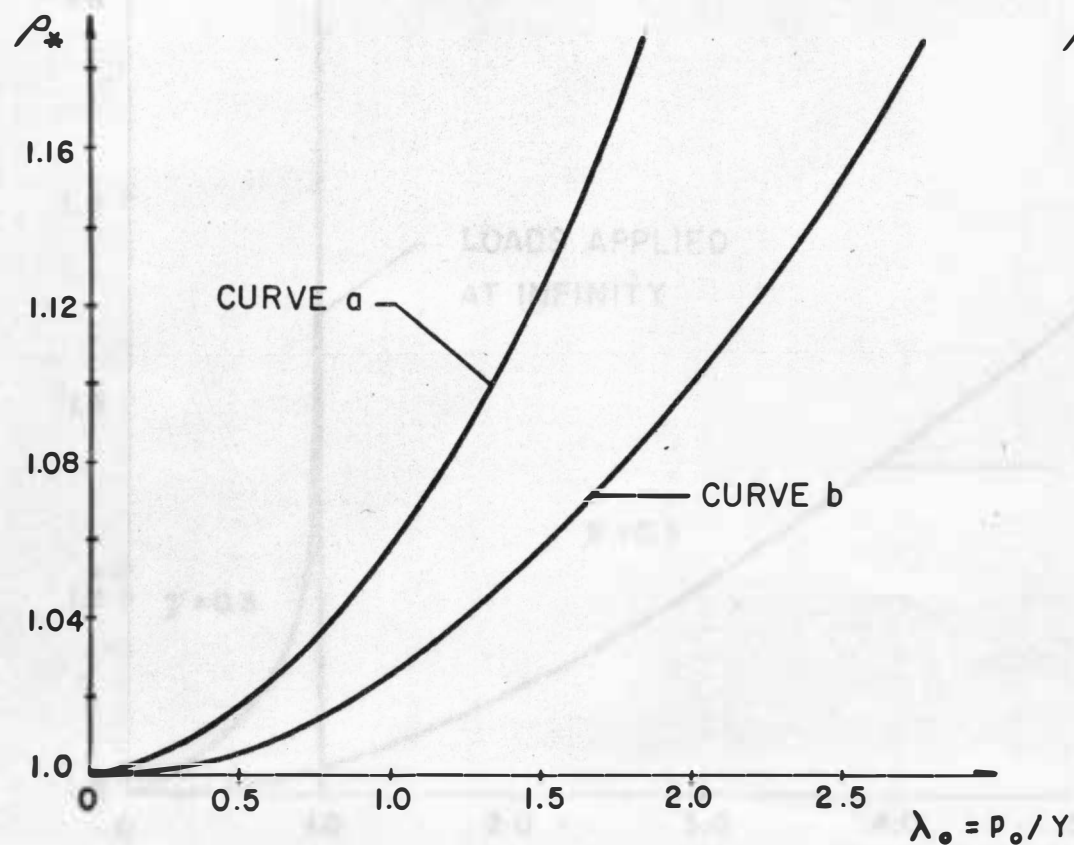


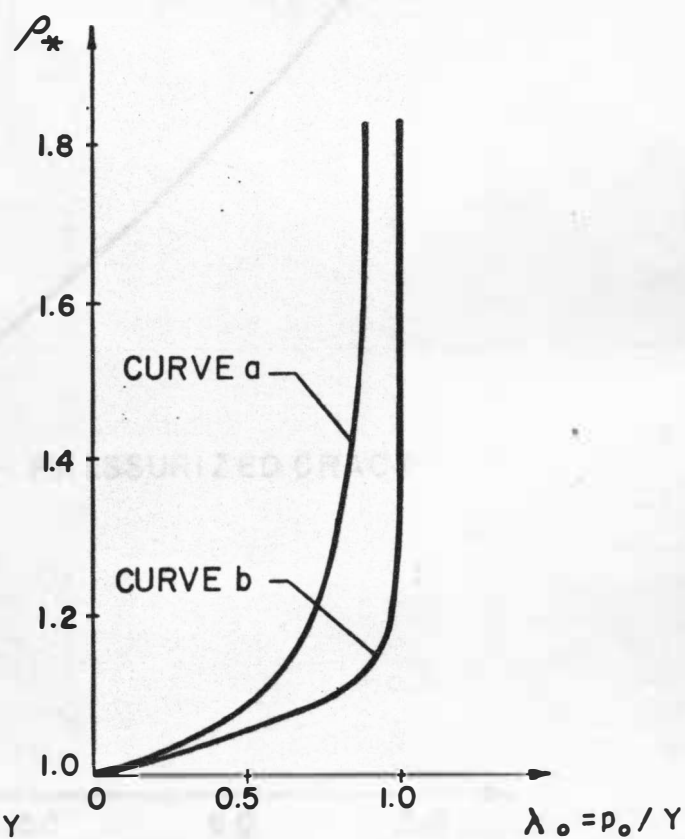
FIG. 11-b SCOPE OF THE PLASTIC ZONE ALONG THE CRACK PLANE.

Loads are applied remotely from the crack:

$$\sigma_r = p_1, \sigma_\theta = p_1, \text{ and } \sigma_z = p_0 \text{ for } r = \infty$$



PRESURIZED CRACK ($\nu = 0.3$)



LOADS APPLIED AT INFINITY ($\nu = 0.3$)

FIG. 12 THE SIZE OF THE PLASTIC ZONE ALONG THE CRACK PLANE ACCORDING TO OLESIAK AND WNUK [12] (CURVE a) AND ACCORDING TO THE PRESENT SOLUTION (CURVE b).

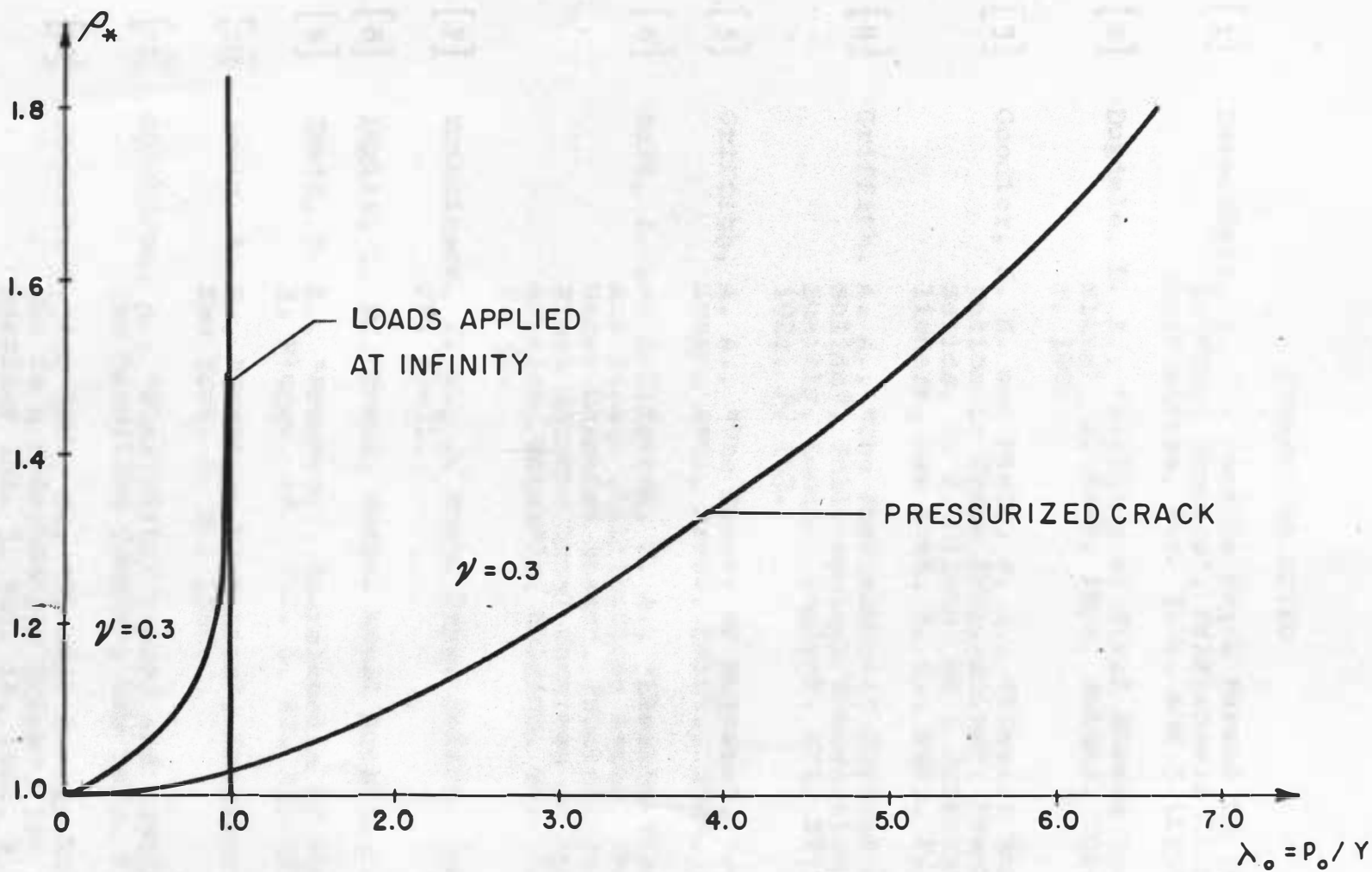


FIG. 13 THE SIZE OF THE PLASTIC ZONE GENERATED IN FRONT OF THE PRESSURIZED CRACK AND IN THE CASE OF LOADS APPLIED REMOTELY FROM THE CRACK.

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